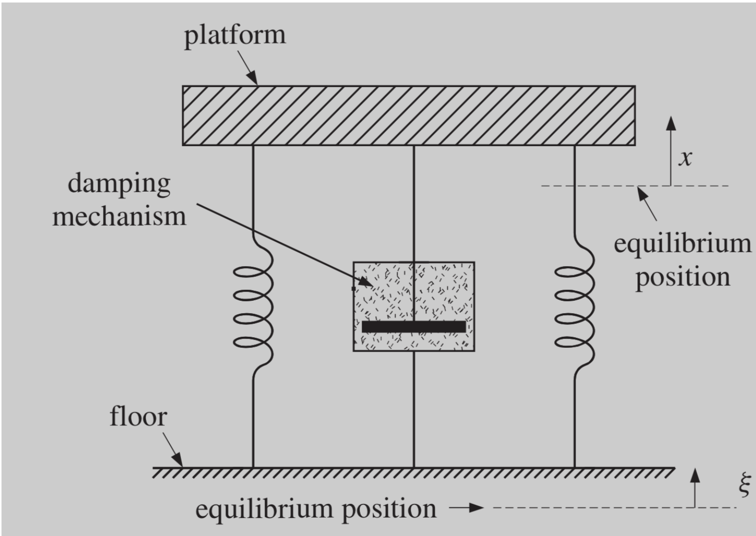
1 My Optical Bench versus the Earthquake (12 points)

Your friend's research lab at Caltech involves doing sensitive length measurements on an optical bench, but being in Southern California, they want to make sure it's isolated from a possible earthquake. In this regard, they ask you to construct a seismic isolation mechanism – good thing you took Ph2a Online!

Here is a schematic diagram of a system that you come up with to isolate the platform holding the optical bench from floor vibrations. The mass of the platform and the optical bench is m (henceforth referred to as *the platform*), the (effective) spring constant of the system is k and there is a damping mechanism (called a dashpot) with damping constant b. The floor is vibrating according to $\xi = \xi(t)$ with respect to its equilibrium position.



(a) [3 points] Let us first write down the equation of motion (Newton's force equation) for the platform. Remember that the damping by the dashpot is proportional to the relative velocity of the platform with respect to the floor.

(b) [1 point] Say that, if the motion of the floor were given by $\xi(t) = \xi_n(t)$ for some function ξ_n then, the steady state motion of the mass is $x(t) = x_n(t)$. Give a short argument or proof that if the motion of the wall were given by $\xi(t) = \sum_n \xi_n(t)$ for some discrete set of functions $\xi_n(t)$, then the steady state solution is $x(t) = \sum_n x_n(t)$

(c) [5 points] During a chat with your seismologist friend at Red Door one night, they mentioned to you that one could model an earthquake of amplitude displacement A which lasts for a duration T as,

$$\xi(t) = \begin{cases} \frac{3A}{T} t & \text{for } 0 \le t \le T/3 \\ \frac{3A}{2}\left(1 - \frac{t}{T}\right) & \text{for } T/3 \le t \le T\,. \end{cases} \tag{1}$$

Focus only on the time interval $0 \le t \le T$ and express this motion in terms of a Fourier sine series (along with determining the expansion coefficients too). Be careful, this is a Fourier series in the time variable but you should be able to write it down based on the general ideas you have learnt in class.
Here is a trig cheat you might find helpful:

$$\int t \sin(t)\, dt = \sin(t) - t\cos(t) + \text{constant}\,. \tag{2}$$

(d) [3 points] Remember how we discussed in class, the steady state solution for a forced, damped oscillator described by oscillation variable $y(t)$ having the equation of motion,

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = \frac{F_0}{m}\cos(\Omega t) \tag{3}$$

is given by,

$$y_{\text{steady}}(t) = \frac{F_0/m}{\sqrt{(\Omega^2 - \omega_0^2)^2 + \gamma^2\Omega^2}} \cos\left(\Omega t - \delta(\Omega)\right) , \tag{4}$$

where,

$$\tan\delta = \frac{\gamma\Omega}{(\omega_0^2 - \Omega^2)} . \tag{5}$$

Armed with this snippet of knowledge, along with your results for parts (b) and (c), what is the steady state motion of the platform when driven by the earthquake? If you were unable to solve part (c), consider a generic Fourier series.

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2 Chasing the Tail [13 points]

In this course, we saw that transverse waves (like a vibrating string) can be thought of as the limit of many masses connected with springs where the displacements are perpendicular to the line of masses (Fig. 1).

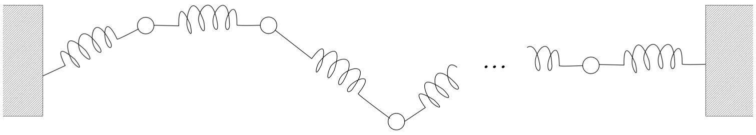

Figure 1: Transverse wave as the limit of finite number of masses

Other waves, like sound waves, are longitudinal where the displacements are along the direction of propagation. In this question, we will investigate the version of this phenomenon with a finite number of masses. To make things more interesting, we will connect the two ends of the chain together to create a loop. Consider three particles of mass m constrained to move along a frictionless ring of radius R, which are connected by springs of stiffness k, also constrained to move along the ring, as shown in Fig. 2.

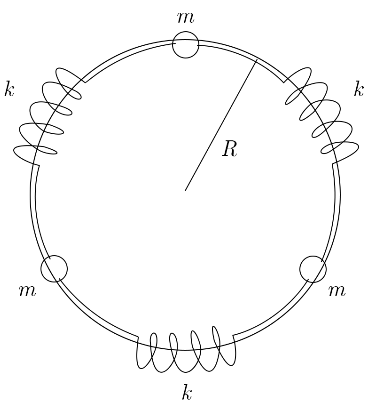

Figure 2: Three masses on a ring of radius R

(a) [1 point] How many normal modes would this system have?

(b) [3 points] Choose angular displacement above equilibrium (i.e. the set $\{\theta_1, \theta_2, \theta_3\}$) as your coordinates for the three particles and determine the equations of motion. First write down your answer as three separate equations, one for each $\ddot{\theta}_j$, $j = 1, 2, 3$ and then express it in matrix form,

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{pmatrix} = -\mathbf{M} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} . \tag{6}$$

Find the 3×3 matrix $\mathbf{M}$ in the above equation.

(c) [3 points] Let us try and find normal coordinates of this system. Recall that for coupled oscillators, normal coordinates are linear combinations of the original coordinates that uncouple the differential equations. For example, the combination $u_1 = (\theta_2 - \theta_3)$ is a normal coordinate such that the equation of motion for u_1 is that of a simple uncoupled harmonic oscillator. Now find the other two normal coordinates u_2, u_3 of the system.
Hint: Try using the three separate equations of motion you wrote down in part (b) to find those linear combinations of $\{\theta_1, \theta_2, \theta_3\}$ *whose equations of motion are decoupled simple harmonic oscillators.*

(d) [3 points] Now that you have the normal coordinates of the system, what are its normal mode frequencies? In case you are not able to identify them right after solving part (c) above, fret not. With some thought you can also figure out that the transformation you found above to get the normal coordinates also gives you the normal mode eigenvectors of the system. For instance, if your linear transformation you found in part (c) to get the normal coordinates is,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathbf{T} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \tag{7}$$

then the eigenvectors corresponding to the 3 columns of the matrix $\mathbf{T}$. If you haven't already, use this information and the matrix $\mathbf{M}$ from part (b) above to find the normal mode frequencies of the system.

(e) [3 points] Draw schematic sketches of the three normal mode oscillations of the system. What does the $\omega = 0$ mode correspond to, physically?

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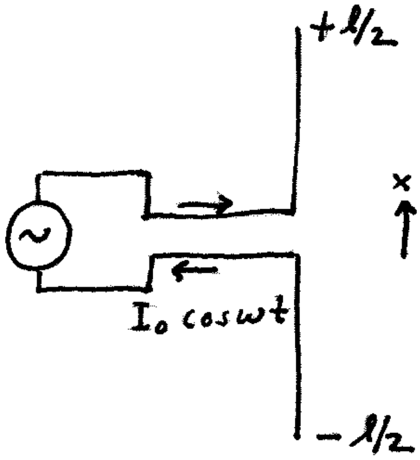
3 Let's Design an Antenna! (12 points)

Right after taking Caltech's Ph2a course online, you set up shop as a wave consultant. Your first client commissions you to design a transmitting antenna to be used in the FM band at 100 MHz, for distances of kilometers.

Well, you know about waves, but aren't too sure about antennas, so you start to doodle with electrical wires and standing waves and all that. Cleverly, you come up with the idea in the figure below:



The antenna part of this is the vertical portion at the right, which can be thought of as a wire of length ℓ cut in the middle, where it is fed with a current from an oscillator.

(a) [3 points] The current in the antenna oscillates due to the driving AC source in such a way that the current at the ends of the antenna is zero (and oscillates in between). Standing waves of current are then set up on the antenna which are responsible for transmission (you can google more later after the exam on how exactly this works). Write down a relationship between the possible wavelengths of standing waves (subject to the boundary conditions) and the length ℓ of the antenna.

(b) [3 points] You want to make your antenna as small as possible, but still be resonant. So you doodle some more and find the longest wavelength standing wave matching your boundary conditions. Make a sketch of the current distribution (at some time t) along the antenna. How long is your antenna (i.e., what is ℓ)?

(c) [3 points] Your client wants to transmit a power of 50 kW. You study your Ph2a EM wave supplementary note, and compute, with more difficulty, the power radiated by an antenna with your geometry and find that, at resonance, it is $P = I_0^2 R/2$, averaged over a cycle, with a resistance $R = 73\ \Omega$, as long as the antenna isn't too near the ground, buildings, etc. [Note: this is the "radiation resistance" of the antenna, relevant to the waves that propagate to long distances. There is also an imaginary, "reactance", term due to the near fields that slightly modifies the optimal antenna length, but we neglect this.] What is the current and what is the voltage required to deliver the desired power? You may neglect all other losses.

(d) [3 points] You are just about to deliver your design, but you check the client's specification again and notice that they want circular polarization. You realize that you have designed an antenna that transmits linearly polarized waves. After some more doodling, mostly with circles, you remember about superposition and get a brilliant idea. Let's just use two of our above antennas, oriented perpendicular to each other, and drive them with a different phase! Make a picture showing this idea. You drive your crossed antenna by splitting the oscillator signal and delivering half to each antenna. You need to change the phase of one of the split signals at its antenna. You do this by adding a length of cable. If the effective index of refraction of your cable is $n = 1.5$, how much cable do you need to add?

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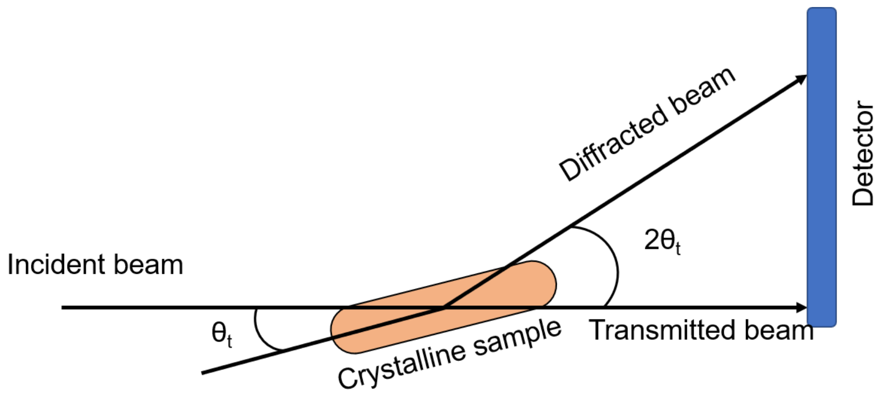
4 Bragging about Deriving Bragg's Law (13 points)

X-Ray Diffraction (XRD), a common materials characterization technique, is based on the physics fundamentals taught in Ph 2a. In essence, the measurement involves shooting x-rays at a crystalline sample with different angles of incidence. When the light hits the atoms in the sample, it undergoes a process called elastic scattering. This results in a reflected beam with an angle twice that of the tilt angle θ_t as shown in the figure below. The pattern of intensity as a function of the sample tilt angle can be used to determine the crystal structure of a material. The interpretation of experiment comes from Bragg's Law, and the focus of this problem will be deriving the Bragg's Law relationship.



(a) [1 point] If the x-ray source is a point source, what shape do the emitted wavefronts have? If this point source is very far away from the crystalline sample, how can you approximate these wavefronts when they are incident on the sample?

(b) [3 points] Now, draw a picture of the incident X-Rays (include their sinusoidal nature), the atoms of a crystal sample, and the diffracted X-Rays.

- Be sure to include θ, the tilt angle, which will be one of our free variables throughout the problem. Also include the angle of the outgoing beam. θ can vary from 0 to 90°
- The x-ray source can be considered to be very far from the sample.
- For the crystalline sample, draw a 2-dimensional slice of a "simple cubic" structure, where the atoms populate the vertices of a lattice made of cubes with side length a.

(c) [3 points] Imagine the beam hitting a single atom on the surface of the sample and its nearest neighbor on the next layer down in the cubic structure. Write down the total difference in the path length the x-rays experience reflecting off two different atoms as a function of a and θ.

(d) [3 points] The signal to the detector is maximized when these beams interfere constructively. Naturally, this depends not only on the material but also the incident wavelength λ. What condition must be met to maximize this signal? (Write an equation with a, θ, λ)

(e) [3 points] XRD instruments commonly use a source of x-rays called Copper $K-\alpha$. This produces x-rays with a wavelength of 1.54 Angstroms (1 Angstrom = 0.1 nm). Polonium is one of the few materials to exhibit a simple cubic lattice structure. If the first ($n = 1$) diffraction peak occurs at $\theta = 27.27°$, what is the lattice spacing? (We are cheating slightly because we're telling you Polonium has a simple cubic structure, but it's a physics class.)
