Ph2a Online: Final Exam

Solutions

1 My Optical Bench versus the Earthquake

(a) The spring force acting on the platform is proportional to the spring extension $(x-\xi)$. The damping force produced by the dashpot is proportional to the relative velocity of the platform with respect to the floor, which is given by $\frac{d}{dt}(x-\xi)$. Thus the equation of motion of the platform is therefore,

$$m\frac{d^2x}{dt^2} = -k(x-\xi) - b\frac{d}{dt}(x-\xi)\,, \tag{1}$$

or,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2\xi + \gamma\frac{d\xi}{dt}\,, \tag{2}$$

where, as usual, $\gamma = b/m$ and $\omega_0^2 = k/m$. One can also rewrite this (though it's not imperative) in terms of a new variable $X = x - \xi$,

$$\ddot{X} + \gamma\dot{X} + \omega_0^2 X = -\ddot{\xi}\,. \tag{3}$$

(b) Any mention of linear superposition of the differential equations involved or a brief exposition of the same is good.

(c) We notice the boundary conditions on $\xi(t)$ as $\xi(0)=0$ and $\xi(T)=0$ and hence we can expand the given $\xi(t)$ function focusing on the time interval $0 \le t \le T$ in a Fourier sine series in the time variable (cosine terms don't contribute since they are not compatible with the boundary conditions),

$$\xi(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{T}\right)\,. \tag{4}$$

One can now invert this using Fourier's inversion trick to get the expansion coefficients B_n's,

$$B_n = \frac{2}{T}\int_0^T dt\xi(t)\sin\left(\frac{n\pi t}{T}\right) = \frac{2}{T}\int_0^{T/3} dt\left(\frac{3A}{T}t\right)\sin\left(\frac{n\pi t}{T}\right) + \frac{2}{T}\int_{T/3}^{T} dt\frac{3A}{2}\left(1-\frac{t}{T}\right)\sin\left(\frac{n\pi t}{T}\right)\,, \tag{5}$$

which after a few lines of tedious integral algebra gives us,

$$B_n = \frac{9A}{\pi^2 n^2}\sin\left(\frac{n\pi}{3}\right) \quad , \quad n = 1,2,3,\cdots\,. \tag{6}$$

(d) We can now plug in our Fourier expansion of ξ in conjuction with the equation of motion from part (a),

$$\ddot{X} + \gamma\dot{X} + \omega_0^2 X = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{T}\right)^2\sin\left(\frac{n\pi t}{T}\right) \equiv \sum_{n=1}^{\infty} C_n\sin\left(\Omega_n t\right)\,, \tag{7}$$

where $C_n = \frac{9A}{T^2}\sin\left(\frac{n\pi}{3}\right)$ and $\Omega_n = n\pi/T$. Using the steady state solution and the principle of linear superposition, we can write the steady state solution for X_{steady},

$$X_{\text{steady}} = \sum_{n=1}^{\infty} \frac{C_n}{\sqrt{(\Omega_n^2 - \omega_0^2)^2 + \gamma^2\Omega_n^2}} \sin\left(\Omega_n t - \delta(\Omega_n)\right) , \tag{8}$$

and correspondingly, the steady state solution for x is,

$$x_{\text{steady}} = X_{\text{steady}} + \xi(t) = \sum_{n=1}^{\infty} \frac{C_n}{\sqrt{(\Omega_n^2 - \omega_0^2)^2 + \gamma^2\Omega_n^2}} \sin\left(\Omega_n t - \delta(\Omega_n)\right) + \sum_{n=1}^{\infty} B_n \sin\left(\Omega_n t\right) , \tag{9}$$

where,

$$\delta(\Omega_n) = \frac{\gamma\Omega_n}{(\omega_0^2 - \Omega_n^2)} . \tag{10}$$

One can also work with the eq. of motion directly in terms of x of Eq. (2) and do a similar steady state solution analysis with linear superposition.

2 Chasing the Tail

(a) There are 3 degrees of freedom, giving 3 normal modes.

(b) Let θ_i be the angular displacements from equilibrium and x_i be the corresponding displacements along the arc. By Hooke's Law, we have

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - x_3) - k(x_1 - x_2) \\ m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3) \\ m\ddot{x}_3 &= -k(x_3 - x_2) - k(x_3 - x_1) \end{aligned}$$

Since $x_i = R\theta_i$, we also have $\ddot{x}_i = R\ddot{\theta}_i$. Substituting, we get

$$\begin{aligned} mR\ddot{\theta}_1 &= -k(R\theta_1 - R\theta_3) - k(R\theta_1 - R\theta_2) \\ mR\ddot{\theta}_2 &= -k(R\theta_2 - R\theta_1) - k(R\theta_2 - R\theta_3) \\ mR\ddot{\theta}_3 &= -k(R\theta_3 - R\theta_2) - k(R\theta_3 - R\theta_1) \end{aligned}$$

After simplifying, we find that

$$\mathbf{M} = \frac{k}{m}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Note that we cannot state from the beginning that $m\ddot{\theta}_1 = -k(\theta_1 - \theta_3) - k(\theta_1 - \theta_2)$, etc. because Hooke's Law applies to linear, not angular, displacements.

(c) We can substitute into the differential equations from part (b) to find that $u_1 = \theta_2 - \theta_3$ satisfies

$$\begin{aligned} \ddot{u}_1 &= \ddot{\theta}_2 - \ddot{\theta}_3 \\ &= -\frac{k}{m}(-\theta_1 + 2\theta_2 - \theta_3) + \frac{k}{m}(-\theta_1 - \theta_2 + 2\theta_3) \\ &= \frac{k}{m}(-3\theta_2 + 3\theta_3) \\ &= -\frac{3k}{m}u_1 \end{aligned}$$

Since the differential equation for u_1 does not depend on any other coordinates, it is uncoupled, so u_1 is a normal mode. Similarly $u_2 = \theta_3 - \theta_1$ satisfies

$$\ddot{u}_2 = -\frac{3k}{m}u_2$$

so it is a normal mode. Now even though $u = \theta_1 - \theta_2$ satisfies $\ddot{u} = -\frac{3k}{m}u$, it is not linearly independent from u_1 and u_2 since $u = -u_1 - u_2$. Therefore, we must find a different normal coordinate. A hint is given to us in part (e) in that there is a normal mode with zero frequency. This corresponds to the three masses rotating on the ring at the same angular velocity, so we should try $u_3 = \theta_1 + \theta_2 + \theta_3$. Indeed, we can calculate that

$$\ddot{u}_3 = 0$$

Another way to find u_3 is to use the fact that normal coordinates correspond to the eigenvectors of **M** and that since **M** is symmetric, there is an orthonormal basis of eigenvectors. An eigenvector orthogonal to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so we have the normal coordinate $u_3 = \theta_1 + \theta_2 + \theta_3$.

(d) The normal frequencies ω_i are given by $\ddot{u}_i = -\omega_i^2 u_i$. We can read off from the uncoupled differential equations in part (c) that

$$\omega_1 = \sqrt{\frac{3k}{m}}$$
$$\omega_2 = \sqrt{\frac{3k}{m}}$$
$$\omega_3 = 0$$

(e) The $\omega_3 = 0$ mode corresponds to the 3 masses rotating at the same angular velocity without compressing or stretching the springs.

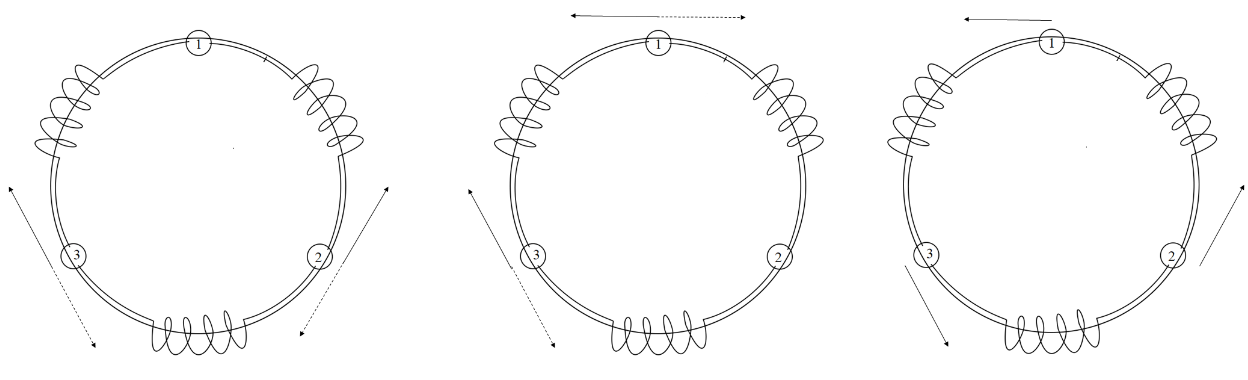


Figure 1: $\omega_1 = \sqrt{\frac{3k}{m}}$ (left); $\omega_2 = \sqrt{\frac{3k}{m}}$ (centre); $\omega_3 = 0$ (right)

3 Let's Design an Antenna!

(a) [3 point] The current in the antenna oscillates due to the driving AC source in such a way that the current at the ends of the antenna is zero (and oscillates in between). Standing waves of current are then set up on the antenna which are responsible for transmission (you can google more later after the exam on exactly how this works), Write down a relationship between the possible wavelengths of standing waves (subject to the boundary conditions) and the length ℓ of the antenna.

Solution: We note that transmission is accomplished with a standing wave on the antenna at the desired carrier frequency. The boundary condition at the ends is that the current must be zero. The current at $x = 0$ is $I_0 \cos \omega t$, so the center of the antenna is an antinode. Thus, the standing wave must be a cosine wave, $I(x, t) = I_0 \cos kx \cos \omega t$. According to the boundary condition, we must have $\cos k\ell/2 = 0$, or $k_n\ell/2 = (2n + 1)\pi/2$, where $n = 0, 1, 2, 3, \ldots$ With $k = 2\pi/\lambda$, this gives the possible wavelengths:

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2}{2n+1}\ell, \quad n = 0, 1, 2, 3, \ldots \tag{11}$$

(b) [3 points] You want to make your antenna as small as possible, but still be resonant. So you doodle some more and find the longest wavelength standing wave matching your boundary conditions. Make a sketch of the current distribution (at some time t) along the antenna. How long is your antenna (i.e., what is ℓ)?

Solution: The longest wavelength we can have that satisfies the boundary conditions has current nodes at the ends and an antinode at the middle, as shown in the sketch below:

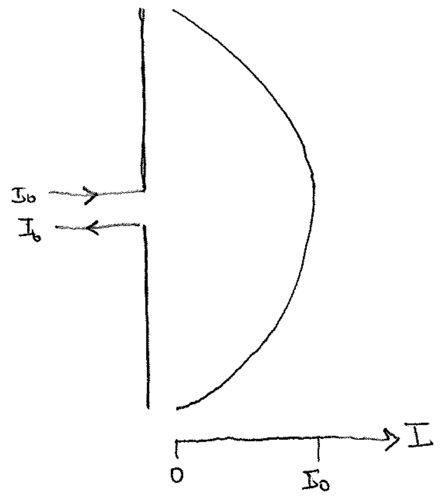


Figure 2: Current distribution along the antenna at $t = 0$, where $x = 0$ at the center, and $x = \pm\ell/2$ at the ends.

The antenna length corresponds to 1 half wavelength of a 100 MHz wave. We assume we are in air, so the index of refraction is close to one and the speed of the wave is the speed of light, $c = 3 \times 10^8$ m/s to a good approximation. Hence, we want an antenna of length

$$\ell = \frac{1}{2}c/f = \frac{1}{2} \cdot \frac{3 \times 10^8}{1 \times 10^8} \text{ m} = 1.5 \text{ m}. \tag{12}$$

(c) [3 points] Your client wants to transmit a power of 50 kW. You study your Ph2a EM wave supplementary note, and compute, with more difficulty, the power radiated by an antenna with your geometry and find that, at resonance, it is $P = I_0^2 R/2$, averaged over a cycle, with a resistance $R = 73\ \Omega$, as long as the antenna isn't too near the ground, buildings, etc. [Note: this is the "radiation resistance" of the antenna, relevant to the waves that propagate to long distances. There is also an imaginary, "reactance", term due to the near fields that slightly modifies the optimal antenna length, but we neglect this.] What is the current and what is the voltage required to deliver the desired power? You may neglect all other losses.

Solution: It should be noted that "resonance" here means our standing wave condition, at least approximately. The current is

$$I_0 = \sqrt{2P/R} = \sqrt{2 \cdot 5 \times 10^4/73}\ \text{A} \approx 40\ \text{A} \tag{13}$$

(or, with a calculator, closer to 37 A).

The voltage is

$$V = I_0 R = 37 \cdot 73\ \text{V} \approx 3000\ \text{V} \tag{14}$$

(or, with a calculator, closer to 2700 V).

We note that we have just computed the "peak" current and voltage; to obtain the rms values we would divide by $\sqrt{2}$.

(d) [3 points] You are just about to deliver your design, but you check the client's specification again and notice that they want circular polarization. You realize that you have designed an antenna that transmits linearly polarized waves. After some more doodling, mostly with circles, you remember about superposition and get a brilliant idea. Let's just use two of our above antennas, oriented perpendicular to each other, and drive them with a different phase! Make a picture showing this idea. You drive your crossed antenna by splitting the oscillator signal and delivering half to each antenna. You need to change the phase of one of the split signals at its antenna. You do this by adding a length of cable. If the effective index of refraction of your cable is $n = 1.5$, how much cable do you need to add?

Solution: Our crossed center-fed dipoles are sketched in the figure below:

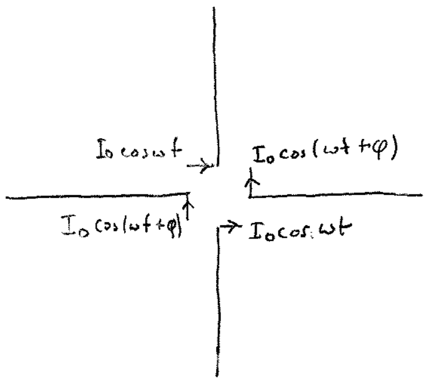


Figure 3: Crossed dipole antenna. To obtain circular polarization, the feed cable to the horizontal antenna is lengthened to give a phase difference of $\phi = \pi/2$.

The phase difference needed to produce a circular polarization is $\phi = \pi/2$, just as with the quarter wave plate and our light beam. Thus, the extra cable length, ΔL we need to add on one of the dipole feedlines is:

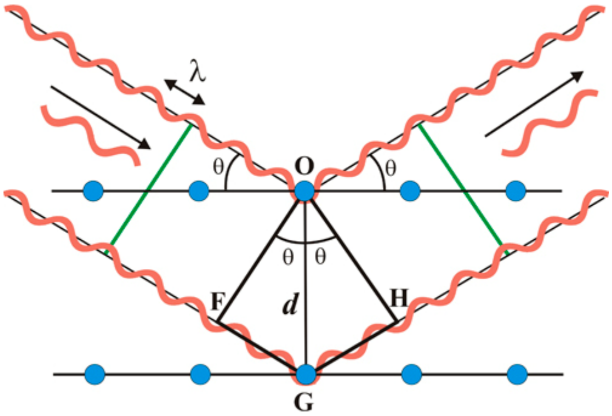
$$\Delta L = \frac{\pi/2}{2\pi}\lambda = \frac{1}{4} \cdot \frac{v}{f} = \frac{1}{4} \cdot \frac{c}{nf} = \frac{2}{3}\frac{\ell}{2} = 0.5\ \text{m}. \tag{15}$$

Note that the wavelength λ here is the wavelength in the cable, not in the air. That is, the wavelength is 2 m in the cable, and 3 m in the air.

4 Bragging about deriving Bragg's Law

1. A point source will emit spherical (or circular) waves. At a sufficient distance from the emitter, the waves can be approximated as planar and in phase. (Half point each for spherical and planar)

2. The correct answer to this section looks something like this:



Of course, if the image is tilted by θ_t that is acceptable as well. One point each for:
angles labeled correctly,
atomic lattice labeled,
sinusodial nature of waves included.

3. The total path length is $2asin(\theta_t)$. Is is shown clearly on the above figure as $2dsin(\theta)$. No partial credit given.

4. Interference is maximized when the diffracting waves are in phase with one another, leading to constructive interference:

$$n * \lambda = 2asin(\theta_t)$$

Due to the geometry of the problem (that θ_t goes from 0 deg to 90 deg), the only valid values for n are the natural numbers ($n = 1, 2, 3, ...$). 1.5 points for indicating the path length difference must be a multiple of the wavelength. 1.5 points for correctly and explicitly indicating $n = 1, 2, 3, ...$.

5. Plug into equation from part c. If the path length difference in part c was incorrect but the substitution was done properly, full credit given. No partial credit given.