Reading: Finish reading Chapter 8 of the textbook, on dispersion.

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34. Some quick questions

(a) We derived in class the diffraction pattern in the far field region for a circular aperture and for small angles $\theta$ from the center of the aperture:

$$|\mathbf{E}\| \propto \left| \frac{a^2}{r} \cdot \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|, \tag{1}$$

where $\mathbf{E}$ is the electric field, $a$ is the radius of the aperture, and $J_1$ is the Bessel function (of the first "kind") of order 1. The first few zeros of $J_1$ are 3.83, 7.02, 10.2, 13.3,..., as may be found tabulated in many places.

In class, we obtained the angle of the first dark ring to be $\theta_1 = 1.22\lambda/2a$. What is the angle of the second dark ring?

(b) You have a small green incandescent lamp with diameter $D$ and wish to experiment with diffraction patterns on a slit of width $d$. How far away from the slit should you put your lamp?

35. A physical model for a dielectric is that electrons in the medium are harmonically bound to the molecules (simple harmonic oscillator at work!!). In the presence of an external electric field, this results in a forced oscillator equation for the electron:

$$m(\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}) = -e\mathbf{E}(\mathbf{x}, t). \tag{2}$$

If the driving force (from the electric field $\mathbf{E}(\mathbf{x}, t)$ in the external wave) is harmonic with frequency $\omega$, we get motion following our familiar resonance formula:

$$\mathbf{x} = -\frac{e\mathbf{E}}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}. \tag{3}$$

When multiplied by the charge $-e$, this gives a time-dependent induced dipole moment. The electric permittivity is created by this dipole moment (we are assuming that magnetic effects are negligible in this discussion).

Things are complicated a little because there may be several ($Z$, say) electrons per molecule, with different resonant frequencies. So, the model for the permittivity is

$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m}\sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\omega\gamma_n}, \tag{4}$$

where $N$ is the number of molecules per unit volume and $\sum_n f_n = Z$. Also, $\epsilon_0$ is the "vacuum permittivity" or "electric constant".

Consider high frequencies, compared with the natural oscillator frequencies. What is the dispersion relation for waves in the dielectric in this limit?

36. We looked at the quantum mechanics of a free particle according to the Schrödinger equation in problem 23. Without putting it in those terms, we derived the dispersion relation and computed the phase and group velocities. The quantum mechanical wave relations for the momentum and energy, $p = \hbar k$ and $E = \hbar\omega$, are thought to hold relativistically, although the Schrödinger equation does not. We already have a relativistic wave equation for light that we derived from Maxwell's equations. This was not a quantum mechanical result, but we could put it in such terms and describe photons.

Let us here consider the relativistic quantum mechanics of a free (and spinless) particle with mass $m$. The appropriate wave equation is called the Klein Gordon equation, but we don't need to actually introduce it here (indeed you could write it down from our discussion of the dispersion relation here). The square of the energy of a free particle of mass $m$ is:

$$E^2 = p^2c^2 + m^2c^4, \tag{5}$$

where $c$ is the speed of light. Note that the energy includes the rest mass of the particle.

(a) What is the dispersion relation for a quatum mechanical relativistic free particle of mass $m$? What are the phase and group velocities? Can they be larger than $c$? If so, is this a problem?

(b) What happens to the phase and group velocities as the momentum gets very large (i.e., in the ultra-relativistc limit)?

(c) What happens to the phase and group velocities in the non-relativistic limit? Do they converge on the results in problem 23? If not, why not?

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