1 Gotham needs your Ph2a skills (10 points)

Batman is looking to revamp his batmobile and one aspect he needs advice on is how to build a good shock-absorber system. In particular, you have been asked to find the required viscous damping coefficient in the system such that it absorbs shocks and returns it to the equilibrium position the fastest.

You model the shock-absorber as a one-dimensional massless spring which is connected to the batmobile's payload of mass m. Recalling from your Ph2a class, you know the 1-D equation of motion of a body of mass m undergoing damped oscillations (with the resistive force proportional to its velocity) is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0\,, \tag{1}$$

or

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0\,, \tag{2}$$

where $\gamma = b/m$ is the damping coefficient and $\omega_0 = \sqrt{k/m}$ is the natural frequency of oscillation.

You remember Ashmeet from Ph2a telling you that a displaced mass returns and settles to the origin the fastest *in the case of critical damping.* Batman won't take your word for it, so you gotta prove it to him.

(a) [3 points] As a warm-up, write down the general solution $x(t)$ for the 3 possible cases of (i.) light damping when $\gamma < 2\omega_0$, (ii.) over/heavy damping when $\gamma > 2\omega_0$ and (iii.) critical damping when $\gamma = 2\omega_0$. In each case, you will of course, have two undetermined constants which would depend on the initial conditions.

(b) [1 point] Let us focus on the overdamped/heavily-damped solution for this part. From your answer in part (a.), write the overdamped solution as,

$$x_{\text{overdamped}}(t) = C_1 \exp\left(-\mu_1 t\right) + C_2 \exp\left(-\mu_2 t\right)\,, \tag{3}$$

with $\mu_1 > \mu_2 > 0$. In the above equation, C_1 and C_2 are unknown constants given by initial condition. In the analysis to follow, you can take both $C_1, C_2 \neq 0$. What are μ_1 and μ_2 in terms of γ and ω_0? Which term in Eq. (3) (the one with μ_1 or μ_2?) is important for late time behavior as $x(t) \to 0$?

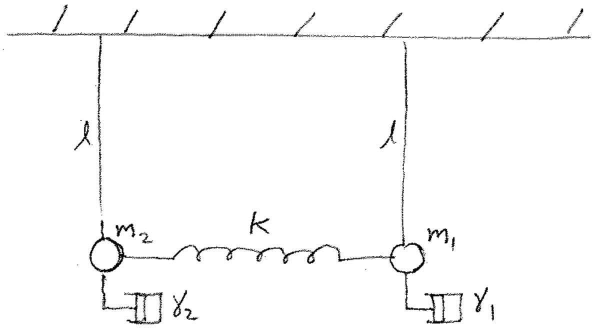
(c) [3 points] Now show that the critically damped oscillator returns and settles to the origin faster than the overdamped case (for the same initial conditions).
Hint: akin to what we did in part (b.) for overdamping, think about the behavior of the oscillator as $x(t) \to 0$ *in the critical damped case.*

(d) [3 points] Okay, now all we are left to do it to demonstrate is that the critical damped oscillator returns and *settles* to the origin even faster than the lightly-damped case too. To do this, again think about how the mass returns to the origin in the critical damped situation and compare with with how the amplitude of the lightly-damped case falls with time.
Hint: remember that exponential suppression envelope in the lightly-damped case?

2 Let's damp together! (15 points)

Coupled oscillators with damping are important in many practical applications, for example, in suppression of earthquake motion in tall buildings. We haven't analyzed such a case yet, although all of our demonstrations have necessarily had damping. To see how it works, let us pursue our coupled pendulum example a bit further. We'll also modify our analysis technique a bit for variety (and perhaps improvement).

Thus, consider the damped, coupled pendulum in the figure below:



We could add sinusoidally varying driving forces, but we'll stick with the transient (homogeneous problem) case for now. To get you started, here is the force equation for mass 1:

$$m_1\ddot{x}_1 + \frac{m_1 g}{\ell}x_1 + k(x_1 - x_2) + m_1\gamma\dot{x}_1 = 0. \tag{4}$$

(a) [2 points] Find matrix D to express the coupled equations of motion as:

$$Dx = 0. \tag{5}$$

You are encouraged to divide your force equations through by m_1 and m_2, as we usually do, before obtaning D. Note that some elements of D will have derviative operators. You can just express these using the notation $d_t \equiv \frac{d}{dt}$ if you wish (and similarly for the second derivative d_t^2).

(b) [3 points] Towards finding the general (homogeneous) solution, we draw from our past experience with this sort of problem and try a solution of the form:

$$x(t) = Ae^{i\omega t}, \text{ where } A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{6}$$

Note that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a vector. Substitute your trial solution into the differential equation $Dx = 0$ and give the result the form of an explicit matrix multiplying A.

(c) [2 points] What is the condition on the matrix you found in part (b), such that there is a non-trivial solution for A (i.e., $A \neq 0$)? Write an explicit equation that could be solved to obtain ω, but don't attempt to solve it (or even multiply out terms).

(d) [3 points] The equation you obtained in part (c) is, I hope, a quartic in ω. Thus, it is possible to solve in closed form. This is of interest, but probably best done using a tool such as Mathematica, which we can't use here. So, instead, let us consider here a subcase, with additional symmetry. Suppose $m_1 = m_2 \equiv m$ and $\gamma_1 = \gamma_2 \equiv \gamma$. With this simplification, go ahead and solve for ω. Express your answer in terms of the eigenfrequencies ω_1 and ω_2 of the undamped coupled pendulum that we determined in class. There should be four solutions in all.

(e) [2 points] Now, for what is to follow, consider very light damping, that is, γ is much less than any characteristic frequency of the system. Find the general solution for the motion, that is $x(t)$. There should be four integration constants corresponding to specifying the initial conditions. Use vectors to indicate the relative motions of masses 1 and 2 in the x coordinate system. Does your answer make sense to you? Hint: before you go to the trouble of solving for eigenvectors, sit back and think. We figured out (and even noted we could guess) the eigenvectors for the undamped coupled pendula in lecture. We have added damping now. Have we disturbed the symmetry of the undamped problem?

(f) [3 points] Suppose that, at $t = 0$, we have the initial conditions (corresponding to an impulse on mass a at $t = 0$):

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \dot{x}(0) = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{7}$$

Find $x(t)$ with these initial conditions. Make sure your units work out correctly.

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3 Sports Station, Weather Station (15 points)

In this problem, we will study how the quality factor plays a role in determining how well a radio can receive signals from certain frequencies without picking up noise from other stations.

You are trying to build a radio that is an LRC circuit consisting of an inductor with inductance L and a variable capacitor with capacitance C in series. Inevitably, there will be some resistance R from the wire, and we will see that it is beneficial for the resistance to be as low as possible. A signal from a radio station will act as an AC power source and will be modeled as a sinusoidal forcing term.

(a) [3 points] Suppose that you are trying to tune your radio to the Lakers game, which is being broadcasted at frequency f and amplitude $\mathcal{E}_0$. Find the general expression for the voltage across the capacitor as a function of time assuming a small value of R. What is the amplitude A of oscillation for large t?

(b) [3 points] For what value of C do we achieve the largest response?

(c) [3 points] What is the width of the peak? Since the natural frequency ω_0 is a function of C and vice versa, you can express your answer in terms of ω_0 instead. That is, how much can you change ω_0 by before the amplitude of response drops to a fraction αA_{max} of the maximum amplitude for a constant α? It may be helpful to make use of linear approximations here to get a cleaner answer.

(d) [3 points] Unfortunately, there is a boring weather station that is broadcasting a signal at a frequency $f' = 1.002f$ very close to the sports station at the same amplitude $\mathcal{E}_0$. Given the two stations broacasting, what is the voltage across the capacitor as a function of time now? Since the radio will be left on, we are only interested in the solution for large t. Linearity of the differential equation may help here.

(e) [3 points] To pick up the most signal from the sports station, you set C, and hence ω_0, to receive the largest (steady state) response A from the sports station as in part (b). However, the radio will also pick up noise of amplitude A' from the weather station. What quality factor is needed in the radio so that noise is at most 1% of the signal? In other words, what does Q have to be so that the $A' \leq 0.01A$? What is the maximum resistance in the circuit you can tolerate?

That's all, folks!
