Problem 20 [10 points $(2+3+2+3)$]

(a) When the far end is free, the reflected pulse will be positive (same as incoming pulse). When the far end is fixed, the reflected pulse will be negative (opposite of incoming pulse). [1 pt for each correct answer]

(b) We have $v^2 = 1/LC$ and for a lossless line $Z^2 = L/C$. We can solve $L = Z/v$ and $C = 1/vZ$. Plugging in $v = 100$ m / 400 ns $= 2.5 \times 10^8$ m/s and $Z = 50\ \Omega$, we find $L = 2 \times 10^{-7}$ H/m and $C = 8 \times 10^{-11}$ F/m. [1 pt for proper method, 1 pt for correct numerical answers, 1 pt for correct units]

(c) The distance between nodes is a half wavelength, so $\lambda = 2$ m. Therefore, $f = v/\lambda = 171.5$ Hz. [2 pts for correct answer]

(d) In short, the particular solution (non-homogeneous) corresponds to the steady-state behavior of the system, while the complementary (homogeneous) solution corresponds to the transient behavior, which is governed by the initial conditions but dies out over time due to damping. [3 pts for correct answer]

Problem 21 [9 Points $(2+5+2)$]

Transverse waves propagate through a string or rope with velocity

$$v = \sqrt{\frac{T}{\mu}}, \tag{1}$$

where μ is the density of the rope and T is its tension. In this case, T is not constant throughout the rope so the speed of the wave will not be constant either—let's see why this is. When the rope is hanging, the forces acting on any given segment of the rope must sum to zero to keep it stationary. If we consider a generic segment of the rope $0 \leqslant y \leqslant h$, there are two relevant forces acting on it: gravity pulls down with a force $m_{\text{segment}} g = \mu h g$ and the point of the rope at h pulls up with the tension T at that point. Thus, we see that

$$T(h) = \mu h g \tag{2}$$

at any height h. Therefore,

$$v(h) = \sqrt{\frac{\mu h g}{\mu}} = \sqrt{hg}. \tag{3}$$

The time taken by the wave to traverse an infinitesimal part of the rope dh at height h is given by $dt(h) = \frac{dh}{v(h)}$, so getting to the top of Millikan will take

$$t = \int_0^{44} \frac{dh}{v(h)} = \int_0^{44} \frac{dh}{\sqrt{hg}} = \frac{1}{\sqrt{g}}[2\sqrt{h}]_0^{44} \approx 4.24\text{sec}. \tag{4}$$

By symmetry, the wave will take just as long to come back down, so the total period of propagation will be roughly **8.5 seconds**.

- -1 for not doubling the time since the pulse went up and down.

Problem 22 [11 Points (6 + 2 + 3)]

(a) At the boundary, the voltage must be transition continuously between the two transmission lines, so the total voltage on the first line ($V_{net}(x,t) = V_I(x,t) + V_R(x,t)$) must equal the voltage on the second line ($V_T(x,t)$) when $x = 0$ and for all time t:

$$V_I(x=0,t) + V_R(x=0,t) = V_T(x=0,t). \tag{5}$$

Each voltage wave has a complex amplitude, a time-dependent phase, and a position dependent phase:

$$V_I(x,t) = V_I e^{i(\omega_I t - k_1 x)},$$
$$V_R(x,t) = V_R e^{i(\omega_R t + k_1 x)} \equiv R V_I e^{i(\omega_R t + k_1 x)},$$
$$V_T(x,t) = V_T e^{i(\omega_T t - k_2 x)} \equiv T V_I e^{i(\omega_T t - k_2 x)}.$$

Let's make sure we understand these equations. Notice that the frequency ω of each wave is a free parameter (that is, before we implement our constraints), but the wavenumber k is determined entirely by the medium through which the wave propagates. The incident wave propagates in $+\hat{x}$ through transmission line 1 and so has wavenumber $-k_1$. The reflected wave propagates through the same line in -$\hat{x}$ and so has wavenumber $+k_1$. The transmitted wave propagates through transmission line 2 in $+\hat{x}$ and so has wavenumber $-k_2$.
Also notice that each "amplitude" in these equations is a complex number which really has a magnitude and a phase of its own (e.g. $V_I = |V_I| \cdot e^{i\phi}$ for some real phase ϕ). These phases are important because without them, every wave satisfying these equations would have a maximum at (x=0, t=0), which certainly doesn't need to be the case. Of course, you could also include this "overall" phase explicitly by writing e.g. $V_I(x,t) = |V_I| e^{i(\omega_I t - k_1 x + \phi)}$.
Anyway, plugging these voltage equations into (5), we find

$$e^{i\omega_I t} + R e^{i\omega_R t} = T e^{i\omega_T t}, \tag{6}$$

which holds for $t = 0$ only when

$$1 + R = T \tag{7}$$

and holds for all time t only when we also have

$$\omega_I = \omega_R = \omega_T \equiv \omega. \tag{8}$$

Now, we also know that current is conserved when it splits from the incident wave to the reflected and transmitted waves, so

$$I_I = I_R + I_T \implies \frac{V_I}{Z_1} = \frac{V_R}{Z_1} + \frac{V_T}{Z_2} \implies \frac{1}{Z_1} = \frac{R}{Z_1} + \frac{T}{Z_2}. \tag{9}$$

- -2 for not explaining how one got the boundary conditions.
- -1 for forgetting current has a direction, and getting the wrong I boundary condition.

(b) We can use equations (7) and (9) to find the reflection and transmission coefficients R and T:

$$\frac{1}{Z_1} = \frac{R}{Z_1} + \frac{1+R}{Z_2} \implies Z_2 = R Z_2 + (1+R) Z_1, \tag{10}$$

or

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \tag{11}$$

from which we gather that

$$T = 1 + R = \frac{2Z_2}{Z_2 + Z_1}. \tag{12}$$

- -1 each for incorrect values of R and T

(c) In the limiting case $Z_2 \to Z_1$, we find R = 0 and T = 1, as expected for a continuous transmission line with unchanging impedance.
In the limiting case $Z_2 \to 0$, we find R = -1 and T = 0. This total reflection of the incident voltage pulse makes sense, since there cannot be a voltage drop across a transmission line with no impedance.
In the limiting case $Z_2 \to \infty$, we find R = 1 and T = 2, which corresponds to maximal transmission. With an infinite impedance on the second transmission line, a finite voltage can easily be maintained by just an infinitesimal current, which is consistent with the high transmission coefficient we derived.

Note: Because the problem was phrased in terms of an incident voltage pulse, it is natural to write the reflection and transmission coefficients in terms of voltage amplitude ratios rather than current ratios (as many of you did), but points were awarded for both answers.

- -0 for incorrect limiting cases (if you took the limit correctly) when your R and T values were wrong
- -0.5 per limiting case if incorrect answer with correct R and T values.
- -0.5 per limiting case if no explanation of why the answers made sense

Problem 23 [10 Points $(6 + 1 + 1 + 2)$]

(a) Note: This answer to part (a) is a somewhat formal derivation; full credit was given for just showing that the appropriate complex exponential wave solves the equation via ansatz (guessing the correct form). The Schrodinger Equation reads

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{x},t) = i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t}. \tag{13}$$

If we consider a separable solution of the form $\Psi(\mathbf{x},t) = X(\mathbf{x})\cdot T(t)$, we find that

$$-\frac{\hbar^2}{2m}T(t)\cdot\nabla^2 X(\mathbf{x}) = i\hbar X(\mathbf{x})\frac{\partial T(t)}{\partial t}. \tag{14}$$

Rearranging this equation, we find that

$$\frac{-\hbar}{2mX(x)}\nabla^2 X(\mathbf{x}) = \frac{i}{T(t)}\frac{\partial T(t)}{\partial t}. \tag{15}$$

Because this equation must be satisfied for all t and $\mathbf{x}$, both sides of the equation must equal a constant (think about this for a little while if you aren't immediately convinced). If we call this constant ω, we see that

$$\frac{\partial T(t)}{\partial t} = -i\omega T(t) \implies T(t) \propto e^{-i\omega t} \tag{16}$$

and

$$\nabla^2 X(\mathbf{x}) = -\frac{2m\omega}{\hbar}X(x) \implies X(x) = Ae^{i\mathbf{k}\cdot\mathbf{x}} + Be^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{17}$$

where we have defined $\mathbf{k}$ such that

$$k^2 = \frac{2m\omega}{\hbar} \tag{18}$$

(this is our dispersion relation). Combining our results, we see that

$$\Psi(\mathbf{x},t) = X(\mathbf{x})\cdot T(t) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + Be^{-i(\mathbf{k}\cdot\mathbf{x}+\omega t)}. \tag{19}$$

Because this expression for Ψ is a function of the quantities ($\mathbf{kx}$ - ωt) and ($\mathbf{kx}$ + ωt), it indeed represents a wave.

For this particular problem, full credit was awarded for just finding one of these solutions, but note for the future that the full solution to any second-order differential equation will always involve a linear combination of two independent solutions.

(b) The speed of the wave is the ratio $\frac{\omega}{k} = \frac{\hbar k}{2m}$ (from Equation 16).

(c) Equation 16 gives $\hbar\omega = \frac{\hbar^2 k^2}{2m}$, so letting $\hbar k$ represent momentum and $\hbar\omega$ represent energy, we have $E = \frac{p^2}{2m}$, which is precisely what we would expect of kinetic energy in classical mechanics.

(d) Letting $\hbar k = p = mv$, we find that

$$\hbar\omega = \frac{mv^2}{2} \implies v = \sqrt{\frac{2\hbar\omega}{m}} = \sqrt{\frac{\hbar^2 k^2}{m^2}} = \frac{\hbar k}{m}, \tag{20}$$

which is a factor of two higher than our answer to part (b). This discrepancy comes from the temporal frequency ω varying with the spatial frequency k: components of the wave with different k will move at different speeds, while the superposition of all these components (called the "wave packet" and representing the actual particle) will diffuse with the (slower) speed of $\frac{\hbar k}{2m}$. You will learn more about these different speeds (known as phase velocity and group velocity) during Weeks 8-9 of Ph 2a.