Problem 24 [10 points]

24.a - 3 points

Let's assume the flute and clarinet are the same length, l. Let's say we have a sound wave $y(x,t)$. The flute has holes on both ends, which means it has open boundary conditions on each end, or that $\frac{dy}{dx} = 0$ at $x = 0$ and $x = l$. The clarinet has a hole at the bottom but a mouthpiece at the top, so it has a closed boundary at the top and an open one at the bottom. Then $y(0,t) = 0$ and $\frac{dy}{dx} = 0$ at $x = l$. Now assume a wave solution $y(x,t) \propto A\cos(kx) + B\sin(kx)$, where we've just dropped the time component since it's not relevant for the problem. For the flute, we require $-kA\sin(kx) + kB\cos(kx) = 0$ for $x = 0$ and $x = l$. Then $B = 0$ and $\sin(kl) = 0$ as a result. That means $k = \frac{n\pi}{l}$ for some integer n. For the clarinet, we require $-kA\sin(kx) + kB\cos(kx) = 0$ at $x = l$, but $A\cos(kx) + B\sin(kx) = 0$ at $x = 0$. Then from the second equation, we have $A = 0$, and from the first, we have $\cos(kl) = 0$. Then $k = \frac{(2n-1)\pi}{2l}$. for some integer n. Note that we have chosen the form of the requirements for k so that the ground state corresponds to $n = 1$. Since for plane waves, the dispersion relation is $k \propto \omega$, then we have $\frac{\omega_{flute}}{\omega_{clarinet}} = \frac{n\pi}{l} * \frac{2l}{(2n-1)\pi} = \frac{2n}{2n-1}$. If we consider $n = 1$, we find that the ratio is 2.

Comments: Many many people only considered the ground state. Only looking at the ground state is a good way to get a feel for the problem, but in this case it gives a poor understanding of the physics going on. In fact, as we go to higher harmonics, the two instruments get closer and closer together.

- 1/3 for only considering the ground state
- 2/3 for an incorrect treatment of higher modes

24.b - 4 points

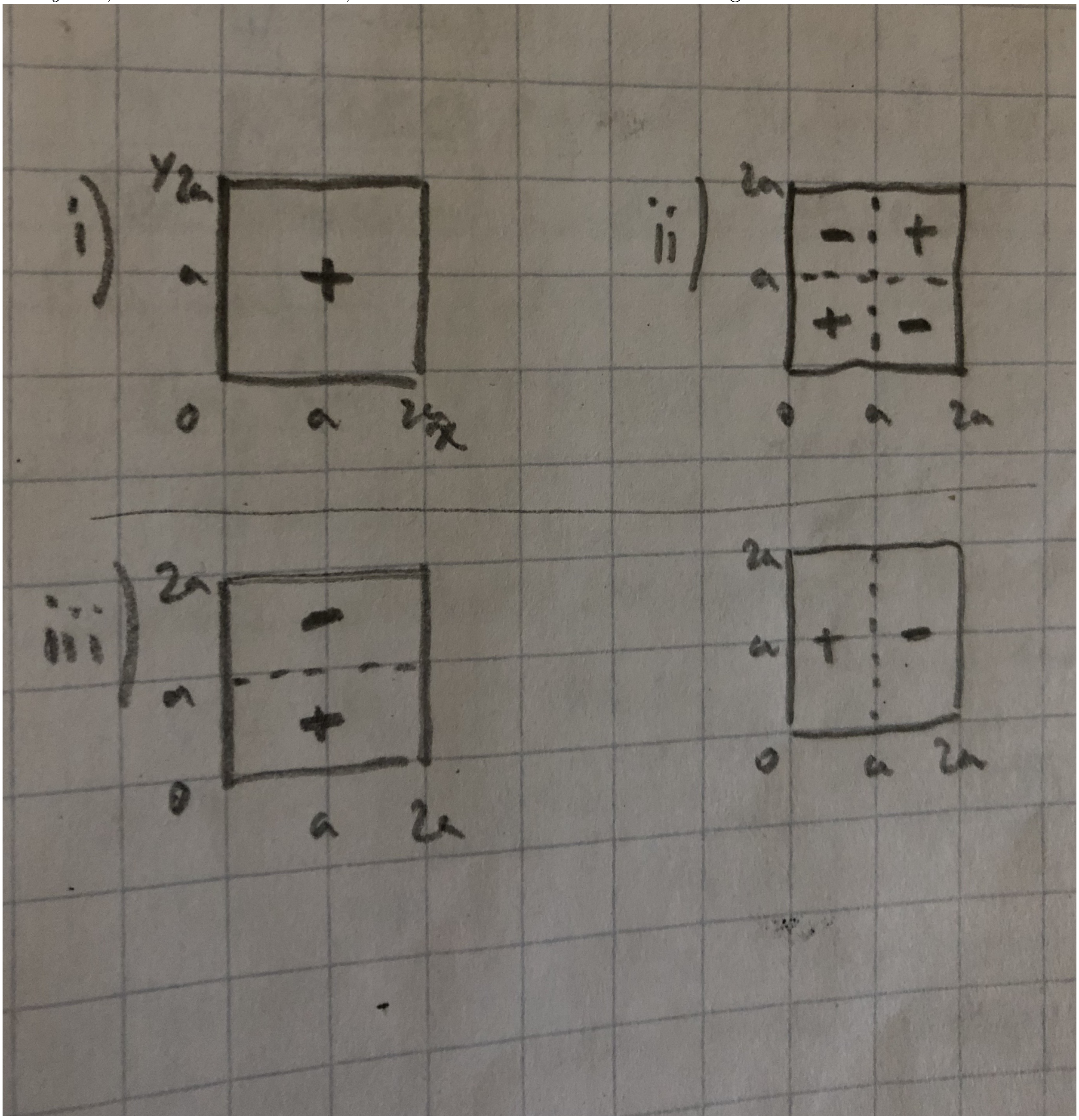
If the intensity of light on the detector is zero in a Michelson interferometer, then we must have tuned the length of the legs such that there is perfect destructive interference when the beams recombine at the beamsplitter. Then there is no light heading from the beamsplitter to the detector. The question we ask ourselves here is: if there was energy being carried through both legs of the interferometer, where did it go? What we have to remember is that the beamsplitter is semi-reflective; that's how the beam gets split in the first place. That means that when the beams in each leg return to the beamsplitter for recombination, only half of each beam gets sent in the direction of the detector. The other half gets sent back to the source, in the exact reverse of the process from which it came! So in reality, we actually have two beams coming out of the interferometer: one going to the detector and one going to the source. If we consider the beam heading to the source, half of that beam will have been reflected twice by the beamsplitter and half will have been transmitted twice, and that difference allows them to pick up a relative phase between them, and it turns out this phase is exactly π. Then the phase difference between the combined beams heading to the detector is opposite that of the combined beams heading to the source. If the detector sees total destructive interference, the source will see total constructive interference, and this is where all the energy goes in our case.

Comments: A lot of people only said there is destructive interference. A good physicist will always follow up with a question about where the energy goes in that case.

- 2/4 for explaining destructive interference well, but not considering energy
- 3/4 for mentioning energy conservation but not explaining it in a complete way

24.c - 3 points

The drumhead (say it has side length $2a$ with the origin in the lower left) has boundary conditions that $z(x,y,t) = 0$ all around the boundary. Then we can look in the x direction and say the normal modes go as $\sin(\frac{n_x \pi x}{2a})$ and the y normal modes go as $\sin(\frac{n_y \pi y}{2a})$ (check to see that these will always obey the boundary conditions!). The lowest nontrivial mode, then, is when $n_x = n_y = 1$. If we define square symmetry to be that the x and y axes are treated equivalently, then we just require $n_x = n_y$ for square symmetry to occur. Then it is clear the next mode with symmetry is $n_x = n_y = 2$, which has nodes at $x = a$ and $y = a$. The lowest modes without $n_x = n_y$ are simply $n_x = 1, n_y = 2$ and $n_x = 2, n_y = 1$. In the first case, there is a node at $y = a$, and in the second case, there is a node at $x = a$. The drawings are attached below.



Comments: Most people did very well on this problem. I also gave credit if you said the $n_x = n_y = 2$ mode was not square symmetric, and instead put the $n_x = n_y = 3$ mode. I did not give credit for saying the $n_x = 2, n_y = 1$ mode was square symmetric, unless there was an excellent description of why this had square symmetry.

- -1 for each incorrect plot, unless one incorrect plot caused another to be incorrect
- 1 point per plot, in general

Problem 25 [10 points]

25.a - 5 points

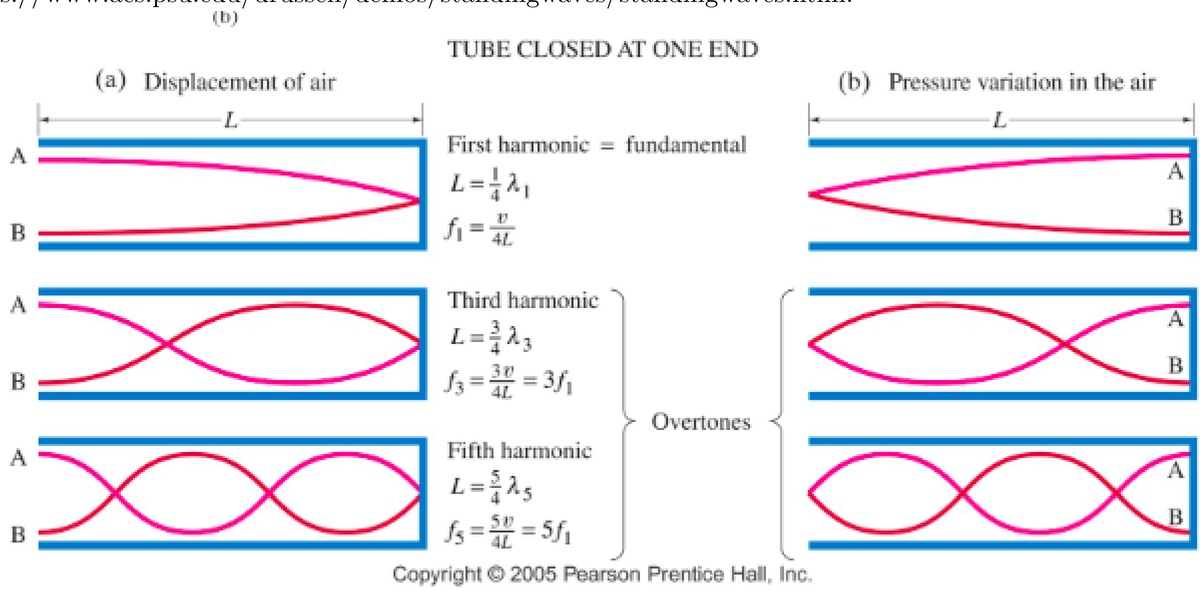
The key to finding the boundary conditions is to note that sound waves propagate as longitudinal displacements of air molecules, which we'll write as $d(x)$. At $x = 0$, our column of air is bounded below by water, which acts as a closed boundary because it is incompressible. The air molecules directly adjacent to the surface of the water cannot oscillate longitudinally because the water blocks their path. Then we have $d(0) = 0$, which is a displacement node. Since the air molecules can't move, but the air molecules above can move, the pressure at the bottom can oscillate freely, so there is a pressure antinode at this point. At $x = L$, we have maximal displacement because the force behind the displacement comes from the source. Thus we have a displacement antinode here, which implies $d'(L) = 0$. We can also discover this result because this end is open and thus the pressure is constant at this end, since it must match atmospheric pressure. The constant pressure means we have a pressure node, which corresponds to the displacement antinode.

Comments: Almost everyone put down good boundary conditions. However, many people said something like "open at the top, closed at the bottom." I was looking for more specificity here, especially since it was not clear if a given person was considering displacement or pressure as their variable (both are wavelike). As such, I did not give full credit unless there were clear equations written. I would also like to warn against a boundary condition like $y(x = L, t) = A_{max}$, where A_{max} is the maximum amplitude. These are waves, so there is oscillation in time. Then A_{max} actually depends on time. It is better to write something like $\frac{dy}{dx} = 0$ at $x = L$, because this is always true.

- 3/5 for no equations and no clear definition of the variable, but correct boundary conditions
- 3/5 for incorrect boundary conditions with good justification
- 4/5 for correct boundary conditions stated in words without accompanying equations

25.b - 5 points

Let's write a generic wave in x to solve for the first two normal modes. Let $d(x,t) = (A\cos(kx) + B\sin(kx))\sin(\omega t + \phi)$. Now let's apply our boundary conditions: $d(0,t) = 0$ and $\frac{d}{dx}d(L,t) = 0$ for all t. From the first we have $A\sin(\omega t + \phi) = 0$ which only holds for all t if $A = 0$. From the second we have $kB\cos(kL)\sin(\omega t + \phi) = 0$ which only holds for all t if $\cos(kL) = 0$. This means we need $kL = \frac{\pi}{2}(2n+1)$ for n a nonnegative integer, so $k = \frac{\pi}{2L}(2n+1)$. Then the lowest two normal modes are $k_1 = \frac{\pi}{2L}$ and $k_2 = \frac{3\pi}{2L}$. We know $k = \lambda^{-1}$ and $\lambda = \frac{2\pi v}{\omega}$, so $k = \frac{1}{2\pi v}\omega$. Then $\frac{\omega_1}{\omega_2} = \frac{k_1}{k_2} = \frac{\frac{\pi}{2L}}{\frac{3\pi}{2L}} = \frac{1}{3}$. A plot of the two normal modes which helps visualize this result is given below, and more information can be found at https://www.acs.psu.edu/drussell/demos/standingwaves/standingwaves.html.



Comments: People did very well on this problem.

- 4/5 for providing a ratio of wavelengths without explicitly stating that $\lambda \propto \omega^{-1}$.

Problem 26 [5 points]

If we follow the derivation on pages 147-8 in the book, we can find the equation for the energy of a normal mode of a string clamped down on both ends, such that its boundary conditions are no oscillation on either end. That system is identical to ours because our weight effectively allows for no movement on the ends of the string. The equation we get is $E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2$, where L is the length of the string, μ is the mass density, A_n is the amplitude of the nth normal mode, and ω_n is the natural frequency of the nth mode, which we know is given by $\frac{n\pi v}{L}$, as that is also given in the book. We are not explicitly given the velocity of waves in the system, but we can find it using equation 5.32 from the book, which shows $v = \sqrt{\frac{T}{\mu}}$, where T is the tension. Substituting these results in yields $E_n = \frac{\mu L A_n^2 n^2 \pi^2 T}{4\mu L^2} = \frac{A_n^2 n^2 \pi^2 T}{4L}$. We are given in the problem that $L = 2m$, $A_n = 5cm$, $n = 10$, and we can calculate $T = mg = 300g * 9.8ms^{-2} = 2.94N$ because the tension is constant throughout the string and must cancel the force of gravity on the weight. Then we find $E_{10} = \frac{(.05m)^2 * 100 * \pi^2 * 2.94N}{4 * 2m} = 0.9J$.

Problem 27 [10 points]

27.a - 7 points

First, we must consider the boundary conditions of the system. At each of the edges, u(x, y, t) = 0 because the edges of the drum are fixed. If we define our origin of the coordinates x, y at the bottom left of the drum, then this gives us the following boundary conditions where x and y are confined to be within 0 and 2a:

$$u(0, y, t) = 0; u(x, 0, t) = 0; u(2a, y, t) = 0, u(x, 2a, t) = 0 \tag{1}$$

We know that the general 2D wave equation is the following:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) \tag{2}$$

From here, there are two good ways to solve this problem: First, we can use separation of variables to say that our solution must take the following form:

$$u(x, y, t) = X(x)Y(y)T(t) \tag{3}$$

Plugging this into the wave equation and then dividing by our solution, we get:

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{v^2} X(x)Y(y)T''(t)$$
$$\rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{v^2} \frac{T''(t)}{T(t)} \tag{4}$$

Let

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda^2 \rightarrow T''(x) + \lambda^2 v^2 T = 0 \tag{5}$$

so that it takes the form of a harmonic oscillator. The negative sign is specifically chosen so that our boundary conditions will hold - we must have a solution that dies away, not one with expontential growth. Given that $w = \lambda v$, we know our solution to this differential equation must be:

$$T(t) = A \cos wt + B \sin wt = A \cos \lambda vt + B \sin \lambda vt \tag{6}$$

Continuing to X, we let

$$\frac{X''(x)}{X(x)} = -\lambda^2 - \frac{Y''(y)}{Y(y)} = -\mu^2 \tag{7}$$

and also:

$$\frac{Y''(y)}{Y(y)} = -\lambda^2 - \frac{X''(x)}{X(x)} = -\alpha^2 \tag{8}$$

Since both of these also harmonic oscillator differential equations, we know that the solutions are the following:

$$X(x) = C \cos \mu x + D \sin \mu x \tag{9}$$
$$Y(y) = E \cos \alpha y + F \sin \alpha y \tag{10}$$

Using our boundary conditions, that X(0) = X(2a) = Y(0) = Y(2a) = 0, we see that since $\cos \mu x = \cos \alpha x = 1$ at x = y = 0, so for those boundary conditions to hold, C = E = 0. Then, we must look at when x = y = 2a. For the sine function to equal zero, the argument must be $2\pi k$ where k is an integer. Thus, we have that $\mu 2a = n\pi$ and $\alpha 2a = m\pi$ We now have that:

$$X(x) = D \sin \big(\frac{n\pi}{2a} x\big) \tag{11}$$
$$Y(y) = F \sin \big(\frac{m\pi}{2a} y\big) \tag{12}$$

We thus have that our normal modes must equal:

$$u(x,y,t) = \sin(\frac{n\pi}{2a}x)\sin(\frac{m\pi}{2a}y)(G_{n,m}\cos w_{n,m}t + H_{n,m}\sin w_{n,m}t) \tag{13}$$

where

$$w_{n,m} = v\sqrt{\alpha^2+\mu^2} = \frac{v\pi}{2a}\sqrt{n^2+m^2}; G_{n,m} = DFA; H_{n,m} = DFB \tag{14}$$

Another way to solve this uses complex exponentials and simplifies from there: A general form solution to the wave equation is:

$$u(x,y,t) = Ae^{i(wt+\phi+\vec{k}\vec{x})} + Be^{i(wt+\phi-\vec{k}\vec{x})} \tag{15}$$

Let us now apply our boundary conditions to this general equation. First we use u(0, 0, t) = 0. This tells us that for the two terms to cancel out for all t and y, A = -B:

$$u(0,y,t) = Ae^{i(wt+\phi)} + Be^{i(wt+\phi)} = Ae^{i(wt+\phi)} - Ae^{i(wt+\phi)} = 0 \tag{16}$$

We next apply the condition that u(2a, 0, t) = 0, which gives us the conclusion that $k_{ym} = \frac{m\pi}{2a}$ where m = 1, 2,... because the sine term will always equal 0 at an integer multiple of π.

$$u(2a,0,t) = Ae^{i(wt+\phi+k_{ym}y)} - Ae^{i(wt+\phi-k_{ym}y)} = 2iAe^{i(wt+\phi)}\sin k_{ym}y = 0 \tag{17}$$

Similarly for y, u(0, 2a, t) = 0 gives us that $k_{xn} = \frac{m\pi}{2a}$ where n = 1, 2,...

$$u(2a,0,t) = Ae^{i(wt+\phi+k_{xn}x)} - Ae^{i(wt+\phi-k_{xn}x)} = 2iAe^{i(wt+\phi)}\sin k_{xn}x = 0 \tag{18}$$

This gives us that for a specific value of n, m, the normal mode is the following where the 2i have been absorbed into $A_{n.m}$. Since $A_{n,m}$ is determined by initial conditions, we have this freedom.

$$u_{n,m}(x,y,t) = A_{n,m}e^{i(w_{n,m}t+\phi_{n,m})}\sin(k_{xn}x)\sin(k_{ym}y) = 0 \tag{19}$$

However, this is not yet physical because it has an imaginary part. Thus, we must take the real part, giving us a single normal mode:

$$u_{n,m}(x,y,t) = A_{n,m}\cos(w_{n,m}t+\phi_{n,m})\sin(\frac{n\pi}{2a}x)\sin(\frac{m\pi}{2a}y) \tag{20}$$

We must now plug this equation back into our wave equation to get our characteristic equation for the system:

$$\frac{\partial^2 u}{\partial t^2} = v^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \tag{21}$$

$$\rightarrow -w^2 A\cos(wt+\phi)\sin(k_x x)\sin(k_y y) \tag{22}$$

$$= v^2\big(-k_x^2 A\cos(wt+\phi)\sin(k_x x)\sin(k_y y) - k_y^2 A\cos(wt+\phi)\sin(k_x x)\sin(k_y y)\big) \tag{23}$$

$$= w_{n,m}^2 = v^2(k_x^2+k_y^2) \rightarrow w_{n,m} = \frac{v\pi}{2a}\sqrt{n^2+m^2} \tag{24}$$

27.b - 3 points

The general solution is simply summing over all possible values of n and m with an arbitrary constant ($A_{n,m}$) multiplied by each term and where $w_{n,m} = \frac{v\pi}{2a}\sqrt{n^2+m^2}$:

$$u(x,y,t) = \sum_{n,m=0}^{\infty}\Bigg(A_{n,m}\cos(w_{n,m}t+\phi_{n,m})\sin(\frac{n\pi}{2a}x)\sin(\frac{m\pi}{2a}y)\Bigg) \tag{25}$$

Problem 28 [15 points]

28.a 4 points

Since our $\psi(x)$ is a wavefunction, it is describing the probability of the wave existing at a point x. However, an infinite potential acts as a insurmountable wall for the wavefunction. Thus, our boundary conditions must be that $\psi(x) = 0$ at x = 0, L, because the probability of finding the particle there must be 0.

28.b 7 points

Within the nucleus where the potential energy is zero, we can use the solutions for a free particle. We remember that the general free particle solution is:

$$\psi(x,t) = Ae^{i(wt+kx)} + Be^{i(wt-kx)} = e^{iwt}(Ae^{ikx} + Be^{-ikx}) \tag{26}$$

However, since we are only interested in a time independent solution, we can ignore the e^{iwt} multiplied by both terms, giving us the following solution:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \tag{27}$$

We must now solve for our free parameters A, B, and k using our boundary conditions. First, we use that ψ(x=0) = 0. As we can see below, for this to hold, A = -B:

$$\psi(0) = Ae^{ik(0)} + Be^{-ik(0)} = A + B = 0 \tag{28}$$

We then use our other boundary condition that ψ(L) = 0. As we can see below, this time for the terms to vanish, kL = nπ where n = 1, 2,... since Sin(x) vanishes when x = nπ.

$$\psi(L) = A(e^{ikL} - e^{-ikL}) = 2iA\sin kL = 0 \tag{29}$$

We now know that our solution takes the form

$$\psi(x) = 2iA\sin\frac{n\pi}{L}x \tag{30}$$

We can now plug this solution back into the Schrodinger equation in order to solve for what E is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) + V(x)\psi(x) = E\psi(x) \tag{31}$$

Since we are only looking at the energy inside the nucleus, V(x) = 0, giving us:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) = E\psi(x) \tag{32}$$

Plugging in our ψ, we get:

$$-\frac{\hbar^2}{2m}(-2ik^2\sin kx) = E2i\sin kx \rightarrow \frac{\hbar^2}{2m}k^2 = E \tag{33}$$

Thus, plugging in our value for k, we get that the energy levels are:

$$E = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}(\frac{n\pi}{L})^2 = \frac{\hbar^2 n^2\pi^2}{2mL^2} \tag{34}$$

where n = 1, 2, 3...

28.c 4 points

We are given that L = 1 fm = $1 \cdot 10^{-15}$ m and m = 1 GeV/c^2= $1{\cdot}10^3$ Mev/c^2. Since $\hbar$ = $6.582{\cdot}10^{-25}$ MeV·s and c = $2.998{\cdot}10^8$ m/s, then we get that the lowest energy level (n = 1) in MeV is

$$E = \frac{\hbar^2 n^2\pi^2}{2mL^2} = 2 * 10^2 MeV \tag{35}$$