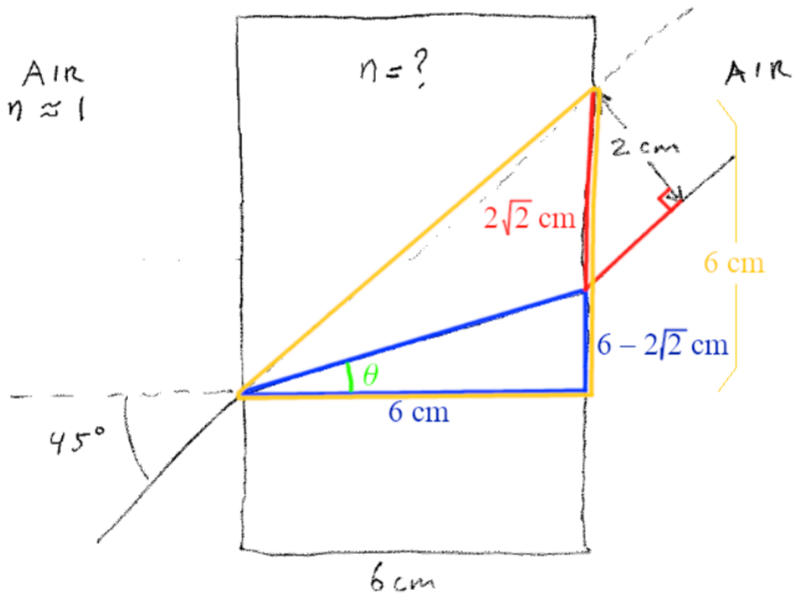
Problem 29 (10 points)

(a) [3 pts]



By doing some geometry, we find that the angle of refraction is

$$\theta = \arctan\left(\frac{6 - 2\sqrt{2}}{6}\right). \tag{1}$$

By Snell's Law, $\sin(45°) = n\sin(\theta)$, and by rearranging these equations we find that

$$\boxed{n = 1.513}. \tag{2}$$

There are many ways to do this problem. You will receive credit as long as your solution is mathematically correct.

(b) [2 pts]

The Rayleigh criterion gives the minimum resolvable angular separation:

$$\Delta\theta = 1.22\frac{\lambda}{D} \tag{3}$$

Plugging in $\lambda = 2.2 \times 10^{-6}$ m and $D = 200$ in $= 5.08$ m, we have

$$\boxed{\Delta\theta = 5.28 \times 10^{-7} \text{ radians}}. \tag{4}$$

Radians are dimensionless, so while indicating the units is preferable, it is not absolutely necessary.

(c) [3 pts]

The full explanation for this problem actually requires you to solve (d) first. Unpolarized light consists of an equal combination of all polarizations. According to (d), when linearly polarized light passes through a polarizer, its intensity is multiplied by $\cos^2(\theta)$ where θ is the angle between the polarization of the light and the axis of the polarizer. The average value of $\cos^2(\theta)$ for $\theta \in [0, 2\pi)$ is

$$\langle \cos^2(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta) dx = \frac{1}{2}. \tag{5}$$

Therefore, the outgoing wave intensity is on average half of the incoming wave intensity:

$$I = \boxed{I_0/2} \tag{6}$$

(d) [2 pts]

The amplitude of the incoming wave was initially $E_0\hat{\mathbf{x}}$. The polarizer allows through all light that is polarized along its axis, and blocks all light that is polarized in the perpendicular direction. The component of $E_0\hat{\mathbf{x}}$ along the axis of the polarizer is $E_0\hat{\mathbf{x}}\cos(\theta)$, so this is the amplitude of the outgoing wave. The intensity is proportional to the square of the amplitude:

$$I = I_0 \left(\frac{E_0 \cos(\theta)}{E_0} \right)^2 = \boxed{I_0 \cos(\theta)^2} \tag{7}$$

Points Distribution

For each subproblem, full points will be awarded for a correct answer and an accompanying work that matches or goes beyond the above solutions. Up to 1 point will be deducted for each wrong answer.

For (c) and (d), the order in which the results are presented is irrelevant, as long as all the information is there or an adequate alternative explanation is provided.

Problem 30 (10 points)

(a) [7 pts]

A wave plate is made of a material that has a different index of refraction depending on the orientation of the light wave passing through it. More specifically, the component of light oscillating in the $\hat{\mathbf{x}}$ direction would travel at a different speed compared to the component of light oscillating in the $\hat{\mathbf{y}}$ direction. In a quarter wave plate, the plate is designed so that the overall phase acquired by the light in the $\hat{\mathbf{y}}$ direction differs from the overall phase acquired by the light in the $\hat{\mathbf{x}}$ direction by $\frac{\pi}{2}$ (each *wave* is 2π of phase, so a *quarter-wave* is $\frac{2\pi}{4} = \frac{\pi}{2}$ of phase).

Writing $\mathbf{p}$ in terms of its components,

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \tag{8}$$

the following matrix introduces a relative phase shift of ϕ to the $\hat{\mathbf{y}}$ component:

$$W_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \tag{9}$$

We are only concerned about relative phase, because an overall phase shift does not affect the polarization of the light.

In the case of a half-wave plate, we saw that $\phi = \pi$. For a quarter-wave plate, $\phi = \pi/2$, and the resulting polarization $\mathbf{p}'$ is given by

$$\mathbf{p}' = W_{\pi/2}\mathbf{p} \tag{10}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}. \tag{12}$$

Therefore,

$$\boxed{\mathbf{p}' = \begin{pmatrix} p_x \\ ip_y \end{pmatrix}}. \tag{13}$$

(b) [2 pts]

If $\mathbf{p} = \hat{\mathbf{x}}$, then we have $p_x = 1$ and $p_y = 0$. Use the result from part (a) to find that

$$\boxed{\mathbf{p}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{\mathbf{x}}}. \tag{14}$$

This is a linear polarization in the $\hat{\mathbf{x}}$-direction.

If $\mathbf{p} = \hat{\mathbf{y}}$, then we have $p_x = 0$ and $p_y = 1$. Use the result from part (a) to find that

$$\boxed{\mathbf{p}' = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\hat{\mathbf{y}}}. \tag{15}$$

This is a linear polarization in the $\hat{\mathbf{y}}$-direction.

(c) [1 pts]

If $\mathbf{p} = \hat{\mathbf{x}}$, then we have $p_x = \frac{1}{\sqrt{2}}$ and $p_y = \frac{1}{\sqrt{2}}$. Use the result from part (a) to find that

$$\boxed{\mathbf{p}' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})}. \tag{16}$$

This is a right circular polarization.

Points Distribution

(a)
- (3 pts): correct interpretation of quarter-wave plate and demonstration of understanding
- (2 pts): correct W_ϕ
- (2 pts): correct answer $\mathbf{p}'$

(b)
- (1 pt): correct answer for $\mathbf{p} = \hat{\mathbf{x}}$, with work shown
- (1 pt): correct answer for $\mathbf{p} = \hat{\mathbf{y}}$, with work shown

(c)
- (1 pt): correct answer, with work shown

Note: It is also possible that the student interpreted the quarter-wave plate to be rotated 90° from the configuration assumed in this solution. In this case, the answers would be the complex conjugates of the ones stated above, and (c) would be a left circular polarization.

Problem 31 (10 points)

(a) [7 pts]

To find the equation for a generic normal mode in this system, we start with the generic waveform

$$f_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t) + B_n \sin(k_n x)\sin(\omega_n t) + \\ C_n \cos(k_n x)\cos(\omega_n t) + D_n \cos(k_n x)\sin(\omega_n t). \quad (17)$$

Since the string is pinned at both ends, we know the normal modes must be zero at $x = 0$ and $x = L$. Thus we cannot have any terms with $\cos(k_n x)$ because that would violate the boundary conditions, so we are left with

$$f_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t) + B_n \sin(k_n x)\sin(\omega_n t). \quad (18)$$

Additionally, we know we are releasing the system from rest, so

$$\frac{d}{dt} f_n(x,0) = 0 \Rightarrow B_n \omega_n \sin(k_n x) = 0 \Rightarrow B_n = 0. \quad (19)$$

Thus our generic normal mode must take the form

$$f_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t). \quad (20)$$

Since we can express any wave as the sum of normal modes, we have

$$f(x,t) = \sum_{n=1}^{\infty} A_n \sin(k_n x)\cos(\omega_n t). \quad (21)$$

Now we will plug in our initial conditions. We can express the initial condition as a Fourier sine series, so that

$$f(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right). \quad (22)$$

Note this is a sum of sines, so we can do a Fourier analysis on it as done on p. 56 of the lecture notes. The result is that

$$A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x,0)dx. \quad (23)$$

We know

$$f(x,0) = \begin{cases} h & L/3 < x < 2L/3 \\ 0 & \text{otherwise.} \end{cases}$$

So we can rewrite our integral as

$$\begin{aligned} A_n &= \frac{2}{L}\int_{\frac{L}{3}}^{\frac{2L}{3}} h \sin\left(\frac{n\pi}{L}x\right) dx = -\frac{2hL}{Ln\pi}\left(\cos\left(\frac{2n\pi}{3}\right) - \cos\left(\frac{n\pi}{3}\right)\right) \\ &= \frac{2h}{n\pi}\left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right). \end{aligned} \quad (24)$$

Then we have solved for A_n. So, we get our final expression for $f(x,t)$ is

$$\boxed{f(x,t) = \sum_{n=1}^{\infty} \frac{2h}{n\pi}\left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right) \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi v}{L}t\right)} \quad (25)$$

where we've subbed in $\omega_n = \frac{n\pi v}{L}$.

(b) [3 pts]

For the wave to return to its initial shape, we solve for

$$f(x,t) = f(x,0). \quad (26)$$

In simpler terms, from (25), we only need to solve for when

$$\cos\left(\frac{n\pi v}{L}t\right) = 1, \quad n \in \mathbb{Z}^+ \tag{27}$$

holds true. (we are interested in the nontrivial solutions, so we ignore the $n = 0$ case.) This means for $\forall n \in \mathbb{Z}^+$, we need some integer m such that

$$\frac{n\pi v}{L}t = 2\pi m. \tag{28}$$

Note that the m's can be different for different n. Rearranging gives our requirement as

$$t = \frac{2L}{v} \times \frac{m}{n}. \tag{29}$$

We must choose an m for each n so that t will be the same for every value of m, which means m/n is a constant. It is easy to see that if we choose $m = n$, then m will be an integer no matter the n, and that $t = 2L/v$. In fact, we can choose $m = 2n$ or $m = 3n$, or m to be any integer multiple of n, and we'll still have m as an integer always and t the same for all n. Thus we **will see** a return to our original shape, and it will happen at

$$\boxed{t = \frac{2L}{v}m, \quad m \in \mathbb{Z}^+.} \tag{30}$$

Points Distribution

(a)
- (1 pts): expression for general waveform
- (3 pts): applying initial and boundary conditions
- (2 pts): deriving the Fourier coefficients
- (1 pt): final expression

(b)
- (1 pt): setting up (26)
- (1 pt): conclusion
- (1 pt): expression for t

Problem 32 (10 points)

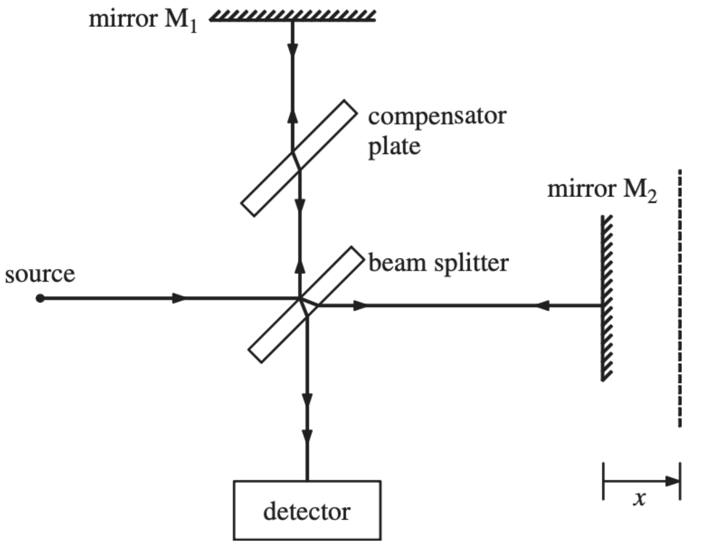


Figure 1: Schematic diagram of the Michelson spectral interferometer. (Fig. 7.8 in King)

We know from the textbook that every cycle of maximum to minimum to maximum intensity in a Michelson interferometer corresponds to a change in distance of $\lambda/2$. In our case, we are not exactly changing the distance, but the change in effective distance corresponds to a change in time taken to traverse the interferometer equivalent to the amount of time it would take light to traverse the effective distance in a vacuum. In math, this means for every cycle, we have a change in time of $(\lambda/2)/c$. Since we observe k cycles, our total change in time is

$$\Delta t = \frac{k\lambda}{2c}. \tag{31}$$

We know the refractive index is defined as

$$n = \frac{c}{v}. \tag{32}$$

So we need to find our new velocity with the He added. Our old velocity is

$$c = \frac{L}{L/c} \tag{33}$$

(distance / time). Our new velocity is

$$v = \frac{L}{\frac{L}{c} + \Delta t} = \frac{Lc}{L + \frac{k\lambda}{2}}. \tag{34}$$

Then

$$n = \frac{c}{v} = \frac{L + k\lambda/2}{L} = 1 + \frac{k\lambda}{2L}. \tag{35}$$

We are given $k = 11$, $\lambda = 633nm$, $L = 0.1m$, and so we get

$$\boxed{n = 1.00003.} \tag{36}$$

Points Distribution

- (3 pts): reasoning for (31)
- (4 pts): derivation of new velocity (34)
- (2 pts): expression for index of refraction
- (1 pt): final numerical value of n

Problem 33 [10 pts]

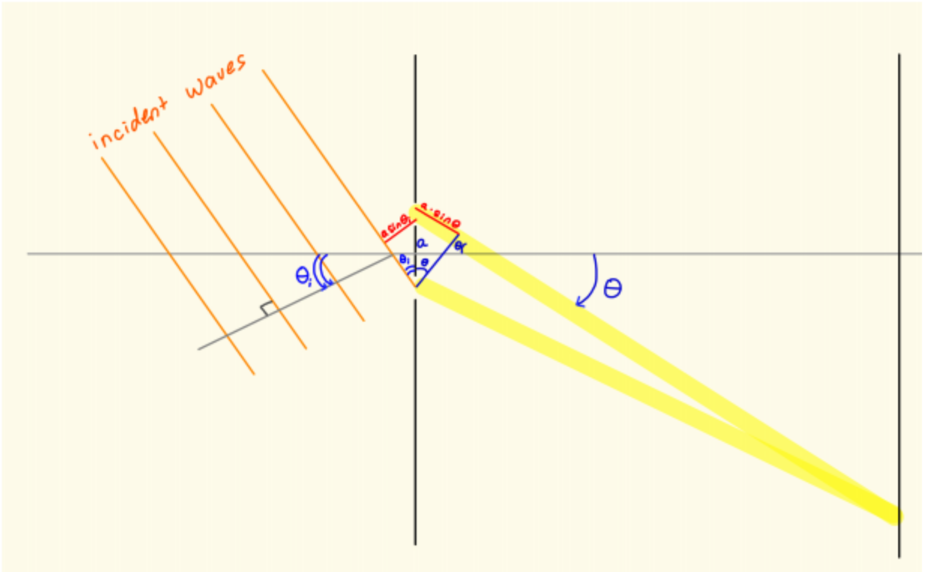


Figure 2: Setup of the system in this problem, defining the angles θ and θ_i and illustrating the difference in path length $a(\sin(\theta)+\sin(\theta_i))$.

(a) [5 pts]

Regardless of the angle of incidence, the dynamics of the wave *beyond* the two slits will be identical, since each slit can be thought of as a "source" of waves. However, *before* the wave reaches the slits, we will observe some different behavior depending on the plane wave's angle of incidence. In particular, if the plane wave approaches the slits at an angle of incidence θ_i as defined in Figure 1, then for a particular wavefront, the distance to the top slit will be greater by $a\ \sin(\theta_i)$ than the distance to the bottom slit. Hence, we will need to adjust the "path length difference" that is used in calculating the average intensity cycle. Previously, this difference was just $a\ \sin(\theta)$; now, it will be $a\ \sin(\theta) + a\ \sin(\theta_i)$ (assuming θ is also defined as in the figure). Therefore, the new average intensity function is

$$\boxed{\bar{I}(\theta) = 2A^2 \cos^2(\frac{ka}{2}(\sin\theta + \sin\theta_i)).} \tag{37}$$

(b) [3 pts]

Maxima of the intensity occur where the argument of the cosine function is an integer multiple of π, i.e.

$$\frac{ka}{2}(\sin\theta + \sin\theta_i) = n\pi \implies \theta = \sin^{-1}(\frac{2n\pi}{ka} - \sin\theta_i), n \in \mathbb{Z}. \tag{38}$$

Recalling that $\lambda = \frac{2\pi}{k}$, we get

$$\boxed{\theta = \sin^{-1}(\frac{m\lambda}{a} - \sin\theta_i), \quad m \in \mathbb{Z}.} \tag{39}$$

(c) [2 pts]

We consider the case where $m = 1$ and θ is small, so

$$\sin\theta \approx \theta = \frac{\lambda}{a} - \sin\theta_i. \tag{40}$$

We see that the angular separation between when $\lambda = 530$ nm and when $\lambda = 660$ nm is given by

$$\Delta\theta = \frac{660 \text{ nm}}{a} - \frac{530 \text{ nm}}{a} = \frac{130 \text{ nm}}{0.01 \text{ mm}} = 0.013 \approx 0.74°, \tag{41}$$

with the red light being diffracted at a larger angle.

Points Distribution

(a)
- (3 pts): justification for the average intensity function
- (2 pts): expression for new average intensity function

(b)
- (2 pts): setting up (38)
- (1 pt): expression for θ

(c)
- (1 pt): small angle approximation
- (1 pt): numerical answer for θ

Overall Comments:

A big theme of this set was that people need to be careful to justify their answers. It is not good practice to pull equations out of thin air, nor is it good practice to write answers without showing your thinking. I took off points if the understanding of the underlying physics was not shown on the page.

30/31: Similar to what I said above, you should at the very least cite the lecture notes when writing down equations without derivation. If you feel you do not understand what polarization actually is, you should definitely talk to a TA and get your questions answered.

32: More mathematical confusion came up here. Since we describe the wave as an infinite series of normal modes, when we look for a return to the original shape, every last one of those normal modes must have the same shape as it had at time zero. Some people calculated a return time in terms of n, which doesn't make sense because n is just an index on the modes. We must have a time that is identical for all n for the return to occur.

33: People did very well on this problem overall. The one common error was double-counting the factor of 2 that you pick up because the light passes through the helium chamber twice. If you are going to use the $\frac{\lambda}{2}$ per cycle adjustment from the book, that is the effective distance change over one pass through the chamber (hence the 2 in the denominator). If you are going to calculate the effective distance change over both passes, then it would be a change of λ per cycle. I'd also like to point out that the answer of 1.00003 is not 1. The 3 at the end is actually significant, and it is remarkable how a Michelson interferometer allows us to discover such a minute difference in refractive index using macroscopic instruments!

Many of you got tripped up on trying to Taylor-approximate the sum in equation (15); however, because the dispersion relation we want is just trying to relate ω and k, we can approximate the ω_n and γ_n terms to *zeroth order*, i.e. we can ignore them completely relative to the large ω^2 term in the denominator.
