Problem 34 (5 points)

a. 3 points

We have already found the first dark ring to be $\theta_1 = \frac{1.22\lambda}{2a}$. We are given $||E|| \propto |\frac{a^2}{r} * \frac{J_1(ka\sin\theta)}{ka\sin\theta}|$. The dark rings occur when there is no E-field because then there is no light. We want the second-smallest zero of the E-field, since we already found the first. It's clear that $ka\sin\theta$ does not blow up, and $\frac{a^2}{r}$ is a constant, so we require $J_1(ka\sin\theta) = 0$. The second zero is given in the problem to occur at 7.02, so then we have $7.02 = ka\sin\theta_2$. We also know the first zero occurs at 3.83, so $3.83 = ka\sin\theta_1, \theta_1 = \frac{1.22\lambda}{2a}$, so $3.83 = ka\sin\left(\frac{1.22\lambda}{2a}\right)$. Then $ka = \frac{3.83}{\sin\left(\frac{1.22\lambda}{2a}\right)}$, so $\sin\theta_2 = \frac{7.02\sin\left(\frac{1.22\lambda}{2a}\right)}{3.83} = 1.83\sin\left(\frac{1.22\lambda}{2a}\right)$. For small angles, $\sin\theta \approx \theta$, so we have $\theta_2 \approx 1.83\frac{1.22\lambda}{2a} = \frac{2.23\lambda}{2a}$.

b. 2 points

There is a good discussion of this in the book on pages 168-169. We require the light to be sufficiently far away from the slit such that the light appears as a coherent source, so that the phase difference from light on the different parts of the slit is small compared to the wavelength. The condition is given on page 169, by relation 7.18: $w << \frac{2l\lambda}{a}$, where here w is the width of the source, l is the distance from the source to the slit, and a is the width of the slit. Converting to our variables, we then get $D << \frac{2l\lambda}{d}$, or, rearranged, $l >> \frac{Dd}{2\lambda}$. For green light, $\lambda \approx 530nm$, so $l >> 1.89 * 10^6 m^{-1} * Dd$.

Problem 35 (10 points)

We are given that

$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m}\sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\omega\gamma_n}. \tag{1}$$

For large ω, we can approximate

$$\omega_n^2 - \omega^2 - i\omega\gamma_n \approx -\omega^2, \tag{2}$$

so

$$\epsilon(\omega) \approx \epsilon_0 - \frac{Ne^2}{m\omega^2}\sum_n f_n = \epsilon_0 - \frac{Ne^2 Z}{m\omega^2}. \tag{3}$$

We now recall that for electromagnetic waves propagating through a dielectric medium,

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu}}, \tag{4}$$

so

$$\omega = \frac{k}{\sqrt{\epsilon\mu}} = \frac{k}{\sqrt{\mu(\epsilon_0 - \frac{Ne^2 Z}{m\omega^2})}}. \tag{5}$$

Rearranging and assuming that $\mu = \mu_0$, we find that

$$\omega^2\mu_0\epsilon_0 - \mu_0\frac{Ne^2 Z}{m} = k^2 \implies \omega^2 = \frac{1}{\mu_0\epsilon_0}(k^2 + \mu_0\frac{Ne^2 Z}{m}). \tag{6}$$

We can write this in a neater form by distributing and using $c^2 = \frac{1}{\mu_0\epsilon_0}$:

$$\omega^2 = c^2k^2 + \frac{Ne^2 Z}{m\epsilon_0}. \tag{7}$$

A note for the curious reader: the quantity $\omega_p \equiv \sqrt{\frac{Ne^2 Z}{m\epsilon_0}}$ actually gives the frequency of oscillation of cold electrons in a plasma (a gas of positive ions separated from lone electrons); this oscillation does not require an externally applied electric field.

Problem 36 (15 points)

a. 5 points

We are given that in the relativistic limit, the square of the energy of a free particle of mass is:

$$E^2 = p^2c^2 + m^2c^4 \tag{8}$$

We are also given that while many non-relativistic equalities do not hold in this limit, the quantum mechanical relationships $p = \hbar k$ and $E = \hbar\omega$ do hold. Our goal in this problem is to find the dispersion relationship for a quantum mechanical, relativistic free particle. In other words, we need a relationship between ω and k and the other parameters in the problem. We can find this by plugging in the given quantum mechanical relationships into the expression for energy, giving us:

$$E^2 = p^2c^2 + m^2c^4 \to (\hbar\omega)^2 = (\hbar k)^2c^2 + m^2c^4 \to \omega^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \tag{9}$$

Solving for ω, we get:

$$\omega = c\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} \tag{10}$$

From this, we can easily find the phase and group velocities using the following relationships:

$$v_p = \frac{\omega}{k} = \frac{c}{k}\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} \tag{11}$$

and

$$v_g = \frac{d\omega}{dk} = \frac{ck}{\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}}} \tag{12}$$

As we can see, the phase velocity can definitely be greater than c, while the group velocity should not ever be greater than c. It makes sense that the phase velocity can actually be greater than c because the phase velocity is just the apparent speed at which the humps of wave appear to travel. So if you interfere two plane waves propagating in opposite directions, but off by a fraction of a degree from being exactly opposite, the interference pattern will produce humps that propagate faster than the speed of light. However, the waves themselves are not travelling faster than light. It is a good thing that we have that the group velocity must always be less than or equal to c, however, since it is the group velocity which corresponds to the information speed. Thus, in order to not violate special relativity, the motion of the wave group, must be less than or equal to c.

b. 5 points

This question is asking us to take the limit as momentum gets very large (the ultra-relativistic limit). Since we have our phase and group velocities in terms of k, not p, we can instead take the limit as k gets very large (since they are just related by a constant). Thus, taking the limit as $k \to \infty$ and keeping $\frac{mc}{\hbar}$ constant ($k >> \frac{mc}{\hbar}$), we get that:

$$v_p = \frac{c}{k}\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} \approx \frac{c}{k}\sqrt{k^2} = c \tag{13}$$

and

$$v_g = \frac{ck}{\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}}} \approx \frac{ck}{\sqrt{k^2}} = c \tag{14}$$

c. 5 points

In the non-relativistic limit, where $p << c$, we get that:

$$E = mc^2(1 + \frac{p^2}{m^2c^2})^{1/2} \approx mc^2(1 + \frac{p^2}{2m^2c^2}) = mc^2 + \frac{p^2}{2m} \tag{15}$$

We can now make the substitutions as before for E and p to find our new dispersion relation:

$$\omega = \frac{mc^2}{\hbar} + \frac{\hbar k^2}{2m} \tag{16}$$

Our new v_p and v_k are:

$$v_p = \frac{\omega}{k} = \frac{mc^2}{\hbar k} + \frac{\hbar k}{2m} \tag{17}$$

and

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} \tag{18}$$

We see that while the group velocity does converge on our previous value of $v_g = \frac{\hbar k}{m}$, the phase velocity does not. It has an extra term that corresponds to the rest energy that was included in this value for the energy and wasn't for the previous calculation in Problem 23.

We can also do the ultra non-relativistic limit $k << \frac{mc}{\hbar}$, and we find that:

$$v_p = \frac{c}{k}\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} \approx \frac{c}{k}\sqrt{\frac{m^2c^2}{\hbar^2}} = \frac{mc^2}{k\hbar} \tag{19}$$

and

$$v_g = \frac{ck}{\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}}} \approx \frac{ck}{\sqrt{\frac{m^2c^2}{\hbar^2}}} = \frac{\hbar k}{m} \tag{20}$$

This gives also gives us the same group velocity, but only the rest mass term from above and completely ignores the regular term from Problem 23 (because it was assumed to be very small).
