Ph2a Online: Midterm Exam

Solutions

1 Gotham needs your Ph2a skills

Part (a)

As always, we guess a solution of the form $x(t) = e^{rt}$. Plugging this into the differential equation, we get

$$\frac{d^2}{dt}\left(e^{rt}\right) + \gamma \frac{d}{dt}\left(e^{rt}\right) + \omega_0^2 e^{rt} = 0 \tag{1}$$

$$r^2 e^{rt} + \gamma r e^{rt} + \omega_0^2 e^{rt} = 0 \tag{2}$$

$$\left(r^2 + \gamma r + \omega_0^2\right) e^{rt} = 0. \tag{3}$$

The solutions to the characteristic equation $r^2 + \gamma r + \omega_0^2 = 0$ are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} \tag{4}$$

which results in two solutions to the differential equation, $e^{r_1 t}$ and $e^{r_2 t}$. The general solution is any linear combination of the two:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{5}$$

For light damping ($\gamma < 2\omega_0$), $\sqrt{\gamma^2 - 4\omega_0^2}$ is imaginary, so r_1 and r_2 are complex. For heavy damping ($\gamma > 2\omega_0$), $\sqrt{\gamma^2 - 4\omega_0^2}$ is real, so r_1 and r_2 are real. For critical damping ($\gamma = 2\omega_0$), the $r_1 = r_2$ so the general solution is a linear combination of e^{rt} and the special term te^{rt}:

$$x(t) = C_1 e^{rt} + C_2 t e^{rt} \tag{6}$$

Part (b)

For heavy damping, the motion is given by

$$x(t) = C_1 \exp\left(\frac{-\gamma - \sqrt{\gamma^2 - 4\omega_0^2}}{2} t\right) + C_2 \exp\left(\frac{-\gamma + \sqrt{\gamma^2 - 4\omega_0^2}}{2} t\right). \tag{7}$$

By matching this with equation (3) in the problem, we have

$$\mu_1 = \frac{\gamma + \sqrt{\gamma^2 - 4\omega_0^2}}{2} \tag{8}$$

$$\mu_2 = \frac{\gamma - \sqrt{\gamma^2 - 4\omega_0^2}}{2}. \tag{9}$$

As specified in the problem, we have chosen the variables such that $-\mu_1$ is more negative than $-\mu_2$, so $\mu_1 > \mu_2$.

Since $\mu_1 > \mu_2$, the term $C_1 \exp(-\mu_1 t)$ decays faster and becomes negligible for $t \to \infty$, and then the solution becomes proportional to $\exp(-\mu_2 t)$.

Part (c)

For critical damping, the motion is given by

$$x(t) = C_1 \exp(-\omega_0 t) + C_2 t \exp(-\omega_0 t)\,. \tag{10}$$

For $t \to \infty$, this solution is proportional to $\exp(-\omega_0 t)$.

In the heavy and critical damping cases, the mass never actually reaches the origin, since the decay is asymptotic; we would like to prove that eventually, the critically damped mass will be closer to the origin than a heavily damped mass with the same initial conditions. This means as $t \to \infty$, we have $\exp(-\omega_0 t) < \exp(-\mu_2 t)$, which is true because

$$\begin{aligned}
\gamma &> 2\omega_0 && (11)\\
4\omega_0\gamma &> 8\omega_0^2 && (12)\\
-4\omega_0^2 &> -4\omega_0\gamma + 4\omega_0^2 && (13)\\
\gamma^2 - 4\omega_0^2 &> \gamma^2 - 4\omega_0\gamma + 4\omega_0^2 && (14)\\
\gamma^2 - 4\omega_0^2 &> (\gamma - 2\omega_0)^2 && (15)\\
\sqrt{\gamma^2 - 4\omega_0^2} &> \gamma - 2\omega_0 && (16)\\
w_0 &> \frac{\gamma - \sqrt{\gamma^2 - 4\omega_0^2}}{2} && (17)\\
w_0 &> \mu_2 && (18)
\end{aligned}$$

The square root operation from (15) to (16) is permissible because $\gamma - 2\omega_0$ is positive.

Part (d)

For light damping, define $i\alpha = \sqrt{\gamma^2 - 4\omega_0^2}$, where α is a real number. The motion is given by

$$\begin{aligned}
x(t) &= C_1 \exp\left(-\frac{\gamma}{2} - \frac{i\alpha}{2}\right) + C_2 \exp\left(-\frac{\gamma}{2} + \frac{i\alpha}{2}\right) && (19)\\
&= \exp\left(-\frac{\gamma}{2}\right)\left[C_1 \exp\left(-\frac{i\alpha}{2}\right) + C_2 \exp\left(\frac{i\alpha}{2}\right)\right]. && (20)
\end{aligned}$$

We recognize this expression as a sinusoid multiplied by a decaying exponential envelope. In light damping, the mass will reach the origin but repeatedly overshoot it; we characterize the speed at which the mass *settles* to the origin by the speed that its amplitude decays, i.e. we look at the exponential factor $\exp(-\gamma/2)$.

Since $\gamma < 2\omega_0$ for light damping, $\gamma/2 < \omega_0$. Therefore, as $t \to \infty$, we have $\exp(-\omega_0 t) < \exp(-\gamma/2)$.

2 Let's damp together!

(a) We are given the force equation for mass 1:

$$m_1\ddot{x}_1 + \frac{m_1 g}{\ell}x_1 + k(x_1 - x_2) + m_1\gamma\dot{x}_1 = 0. \tag{21}$$

The other force equation is:

$$m_2\ddot{x}_2 + \frac{m_2 g}{\ell}x_1 + k(x_1 - x_2) + m_1\gamma\dot{x}_1 = 0. \tag{22}$$

Then (dividing these equations first by m_1 and m_2) matrix D is:

$$D = \begin{pmatrix} d_t^2 + \frac{g}{\ell} + \frac{k}{m_1} + \gamma_1 d_t & -\frac{k}{m_1} \\ -\frac{k}{m_2} & d_t^2 + \frac{g}{\ell} + \frac{k}{m_2} + \gamma_2 d_t \end{pmatrix}. \tag{23}$$

(b) Now we try a solution of the form:

$$x(t) = Ae^{i\omega t}, \text{ where } A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{24}$$

Substituting in to the differential equation $Dx = 0$ gives:

$$\begin{pmatrix} -\omega^2 + \frac{g}{\ell} + \frac{k}{m_1} + i\gamma_1\omega & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + \frac{g}{\ell} + \frac{k}{m_2} + i\gamma_2\omega \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0. \tag{25}$$

Note tht this is no longer either a real nor a symmetric matrix. But plunge ahead.

(c) What is the condition on the matrix you found in part (b), such that there is a non-trivial solution for A (i.e., $A \neq 0$)? Note that if we have a matrix equation $Mx = 0$, and if M has an inverse M^{-1}, then we could multiply by M^{-1} and find:

$$x = M^{-1}Mx = 0. \tag{26}$$

Therefore, for x to have a non-trivial solution, M must be singular, $\det M = 0$.

Thus, $\det D = 0$, and we obtain

$$\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m_1} + i\gamma_1\omega\right)\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m_2} + i\gamma_2\omega\right) - \left(\frac{k}{m_1}\right)\left(\frac{k}{m_2}\right) = 0. \tag{27}$$

(d) Consider here a subcase, with additional symmetry. Suppose $m_1 = m_2 \equiv m$ and $\gamma_1 = \gamma_2 \equiv \gamma$. We have now

$$\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m} + i\gamma\omega\right)\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m} + i\gamma\omega\right) - \left(\frac{k}{m}\right)^2 = 0. \tag{28}$$

for ω. This is much easier, giving the quadratic pair of equations

$$-\omega^2 + \frac{g}{\ell} + \frac{k}{m} + i\gamma\omega = \pm\frac{k}{m}. \tag{29}$$

Solving these two quadratic equations, and using

$$\omega_1^2 = g/\ell \qquad \text{and} \qquad \omega_2^2 = g/\ell + 2k/m, \tag{30}$$

we obtain the four solutions for ω:

$$\begin{aligned} \omega'_{1\pm} &= i\frac{\gamma}{2} \pm \sqrt{\omega_1^2 - \left(\frac{\gamma}{2}\right)^2} \\ \omega'_{2\pm} &= i\frac{\gamma}{2} \pm \sqrt{\omega_2^2 - \left(\frac{\gamma}{2}\right)^2} \end{aligned} \tag{31}$$

Express your answer in terms of the eigenfrequencies ω_1 and ω_2 of the undamped coupled pendulum that we determined in class. There should be four solutions in all.

(e) Noting that the only addition to the problem without damping is a symmetric damping on both masses, the normal coordinates still work (the only difference is that now the motion is damped), the general solution for the motion is thus:

$$x = e^{-\gamma t/2}\left\{B\cos\left[\sqrt{\omega_1^2-\left(\frac{\gamma}{2}\right)^2}t+\phi_1\right]\begin{pmatrix}1\\1\end{pmatrix}+C\cos\left[\sqrt{\omega_2^2-\left(\frac{\gamma}{2}\right)^2}t+\phi_2\right]\begin{pmatrix}1\\-1\end{pmatrix}\right\}. \tag{32}$$

The four integration constants are B, C, ϕ_1, ϕ_2.

(f) At $t=0$ we have the initial conditions:

$$x(0)=\begin{pmatrix}0\\0\end{pmatrix}, \qquad \dot{x}(0)=\begin{pmatrix}v\\0\end{pmatrix}. \tag{33}$$

The initial condition on the position, $x(0)=0$ is satisfied by letting $\phi_1=\phi_2=\pm\pi/2$. Thus, we may write:

$$x(t) = e^{-\gamma t/2}\left\{B\sin\left[\sqrt{\omega_1^2-\left(\frac{\gamma}{2}\right)^2}t\right]\begin{pmatrix}1\\1\end{pmatrix}+C\sin\left[\sqrt{\omega_2^2-\left(\frac{\gamma}{2}\right)^2}t\right]\begin{pmatrix}1\\-1\end{pmatrix}\right\}. \tag{34}$$

We differentiate to get the velocity:

$$\dot{x}(t) = -\frac{\gamma}{2}x(t)+e^{-\gamma t/2}\left\{B\alpha\cos\alpha t\begin{pmatrix}1\\1\end{pmatrix}+C\beta\cos\beta t\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \tag{35}$$

where we have for convenience defined

$$\alpha\equiv\sqrt{\omega_1^2-\left(\frac{\gamma}{2}\right)^2} \qquad \beta\equiv\sqrt{\omega_2^2-\left(\frac{\gamma}{2}\right)^2}. \tag{36}$$

Then, at $t=0$, we have

$$\dot{x}(0)=\begin{pmatrix}v\\0\end{pmatrix}=B\alpha\begin{pmatrix}1\\1\end{pmatrix}+C\beta\begin{pmatrix}1\\-1\end{pmatrix}. \tag{37}$$

Therefore,

$$B=\frac{v}{2\alpha} \qquad \text{and} \qquad C=\frac{v}{2\beta}. \tag{38}$$

Then the solution for $x(t)$ satisfying the boundary conditions is:

$$x(t)=\frac{v}{2}e^{-\gamma t/2}\left\{\frac{\sin\left[\sqrt{\omega_1^2-\left(\frac{\gamma}{2}\right)^2}t\right]}{\sqrt{\omega_1^2-\left(\frac{\gamma}{2}\right)^2}}\begin{pmatrix}1\\1\end{pmatrix}+\frac{\sin\left[\sqrt{\omega_2^2-\left(\frac{\gamma}{2}\right)^2}t\right]}{\sqrt{\omega_2^2-\left(\frac{\gamma}{2}\right)^2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}. \tag{39}$$

3 Sports Station, Weather Station

a)

We are given an LRC circuit with the signal from a radio station acting as an AC power source. By Kirchhoff's Loop Rule, and using the voltage relations for individual components $V_R=IR$, $V_L=-L\frac{dI}{dt}$, $V_C=\frac{q}{C}$, we can sum the voltages to find $\frac{q}{C}+L\frac{dI}{dt}+RI=\epsilon_0\cos(\omega t)$, where ϵ_0 is the amplitude of the signal and $\omega=2\pi f$.

We want an expression just for V_C, so we can substitute $V_C = \frac{q}{C}$ and $\dot{q} = I$ to get the differential equation $V + LC\ddot{V} + RC\dot{V} = \epsilon_0 \cos(\omega t)$. Rearranging yields $\ddot{V} + \gamma\dot{V} + \omega_0^2 V = f_0 \cos(\omega t)$, where $\gamma = \frac{R}{L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$, $f_0 = \frac{\epsilon_0}{LC} = \epsilon_0\omega_0^2$.

If R is small, then we have an underdamped system. Referring to section 3.2.2 of the text and the light damping solution eq. 2.8, we find the general solution to be $V(t) = Be^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi) + A\cos(\omega t - \delta)$, where $\omega_1 = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}$, $A = \frac{\epsilon_0\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}}$, $\tan(\delta) = \frac{\omega\gamma}{\omega_0^2-\omega^2}$, B, ϕ some constants determined by initial conditions. Notice this solution's first term decays away and is derived from the solution to the light damping case, while the second term is the long-term performance of the circuit after the decaying term dies out. Thus for large t, the amplitude is just A, which is defined above.

b)

At large t, we just want to find the maximum of A with respect to C to determine the place of maximum response. Looking at $A = \frac{\epsilon_0\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}}$, we can see that only ω_0 depends on C. If we write $x = \omega_0^2 = \frac{1}{LC}$, then we can maximize A^2 with respect to x and deduce the optimal C from that.

$\frac{d}{dx}A^2 = \frac{d}{dx}\frac{\epsilon_0^2 x^2}{(\omega^2-x)^2+\omega^2\gamma^2} = \frac{2\epsilon_0^2 x((\omega^2-x)^2+\omega^2\gamma^2)+2\epsilon_0^2 x^2(\omega^2-x)}{((\omega^2-x)^2+\omega^2\gamma^2)^2}$. Setting this equal to zero yields $2\epsilon_0^2 x((\omega^2 - x)^2 + \omega^2\gamma^2) + 2\epsilon_0^2 x^2(\omega^2 - x) = 0$ or $\omega^4 - \omega^2 x + \omega^2\gamma^2 = 0$ or $x = \omega^2 + \gamma^2$. Then $\frac{1}{LC} = \omega^2 + \gamma^2$.

For small R, we also have small γ, which would indicate $\frac{1}{\omega^2+\gamma^2} \approx \frac{1}{\omega^2} = LC$. Then $C \approx \frac{1}{L\omega^2}$ is the optimal value of C for maximum response. Note our approximation that $\omega^2 + \gamma^2 \approx \omega^2 = \frac{1}{LC}$ implies $\omega \approx \omega_0$ when ω_0 is set at the maximum response.

c)

Since we're operating at maximum response, we can make the approximation that $\omega_{0,max} \approx \omega$. The maximal amplitude, then, is $\frac{\epsilon_0\omega}{\gamma}$, which is found just by plugging in our approximation to the formula from part a for A. Then we want to find what frequency ω_0 such that the amplitude at this frequency is $\alpha A_{max} = \frac{\alpha\epsilon_0\omega}{\gamma}$. Using our expression for A, we can write $\frac{\epsilon_0\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}} = \frac{\alpha\epsilon_0\omega}{\gamma}$.

Now we'll square both sides and simplify to get $\gamma^2\omega_0^4 = \alpha^2\omega^2((\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2)$ or $(\omega^2 - \omega_0^2)^2 = \frac{1}{\alpha^2}\frac{\omega_0^4\gamma^2}{\omega^2} - \omega^2\gamma^2 = (\frac{1}{\alpha^2}(\frac{\omega_0}{\omega})^4 - 1)\omega^2\gamma^2$. We now recognize that the quality factor Q is proportional to R^{-1}, and so since we have small R we have high Q, and thus this peak is tight. Then we can approximate $\frac{\omega_0}{\omega} \approx 1$. Thus we have $(\omega^2 - \omega_0^2)^2 \approx (\frac{1}{\alpha^2} - 1)\omega^2\gamma^2$ or $\omega^2 - \omega_0^2 \approx \pm\sqrt{\frac{1}{\alpha^2} - 1}\omega\gamma$.

Then we can solve for ω_0 and we get $\omega_0 \approx \sqrt{\omega^2 \pm \sqrt{\frac{1}{\alpha^2} - 1}\omega\gamma} = \omega\sqrt{1 \pm \sqrt{\frac{1}{\alpha^2} - 1}\frac{\gamma}{\omega}}$. But we know γ is small, so we can approximate the outer square root. Then $\omega_0 \approx \omega(1 \pm \frac{1}{2}\sqrt{\frac{1}{\alpha^2} - 1}\frac{\gamma}{\omega})$. Since we started at ω, then $\Delta\omega_0 \approx \gamma\sqrt{\frac{1}{\alpha^2} - 1} \approx \frac{\omega}{Q}\sqrt{\frac{1}{\alpha^2} - 1}$, and this is the change in ω_0 required to decrease the amplitude by a factor of α.

(d)

Given the existence of a boring weather station with a broadcasting signal frequency $f' = 1.002f$ with the same amplitude ϵ_0 s the sports station, we can compute the voltage across the capacitor as a function of time via a second-order ODE that takes on a similar form as the one used in part (a):

$$\ddot{V} + \gamma\dot{V} + \omega_0^2 V = f_0 \cos\omega t + f' \cos(\omega' t + \theta) \tag{40}$$

where $\omega' = 2\pi f', \gamma = \frac{R}{L}, \omega_0 = \frac{1}{\sqrt{LC}}, f_0 = \frac{\epsilon_0}{LC} = \epsilon_0\omega_0^2$ and we use linear theory for the approximation.

Note the familiar form of this ODE. We can solve it using our previous knowledge of the general form of such an ODE and yield the following solution (Similar process of deriving the general form was used in part(a) and also in the derivation of eq. 2.8.). The solution to the ODE takes the following form once we continue to use the solution derived in part(a) and consider the additional source term in the ODE:

$$V(t) = Ce^{-\gamma t/2}\cos(\omega' t + \phi) + A\cos(\omega t - \delta) + A'\cos(\omega' t + \theta - \delta') \tag{41}$$

where we have

$$A' = \frac{f_0}{\sqrt{(\omega'^2 - \omega_0^2) + \omega'^2\gamma^2}} \tag{42}$$

$$\tan\delta' = \frac{\omega'\gamma}{\omega_0^2 - \omega'^2}. \tag{43}$$

We can similarly analyze the solution by considering that the first term decays away and the rest is simply derived from the light damping case, while the second and third term shows how the RLC circuit behaves even after the first term vanishes at large t.

(e)

We will start off by considering what quality factor we need for the noise to be at most 1% of the signal using our results form part(c):

$$\Delta\omega_0 \approx \frac{\omega}{2Q}\sqrt{\frac{1}{\alpha^2} - 1} \tag{44}$$

This will naturally lead to the following calculations of the quality factor needed for this case:

$$\frac{\Delta\omega_0}{2} = \omega' - \omega \tag{45}$$

$$\frac{\omega}{2Q}\sqrt{\frac{1}{0.01^2} - 1} = 0.002\omega \tag{46}$$

$$\Rightarrow Q = \frac{99}{0.004} \tag{47}$$

$$\approx 25000 \tag{48}$$

Now, we know what Q value we need, we can now shift our focus onto the RLC circuit provided:

$$Q = \frac{\omega_0}{\gamma} \tag{49}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{50}$$

$$= \frac{L\omega}{R} \tag{51}$$

Therefore, we have $R \leq \frac{L\omega}{Q} \approx \frac{L\omega}{25000}$ as desired.