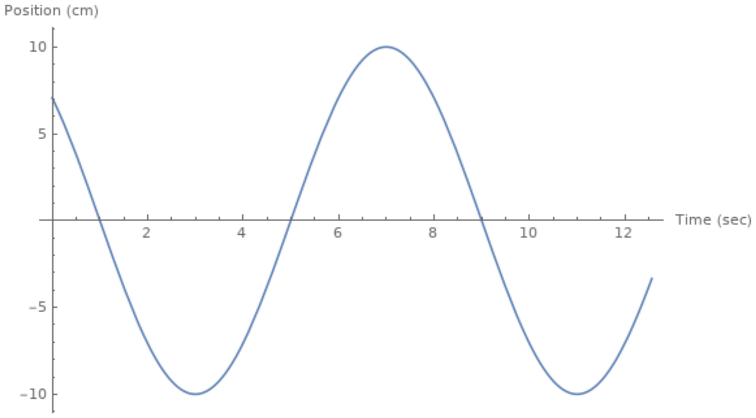
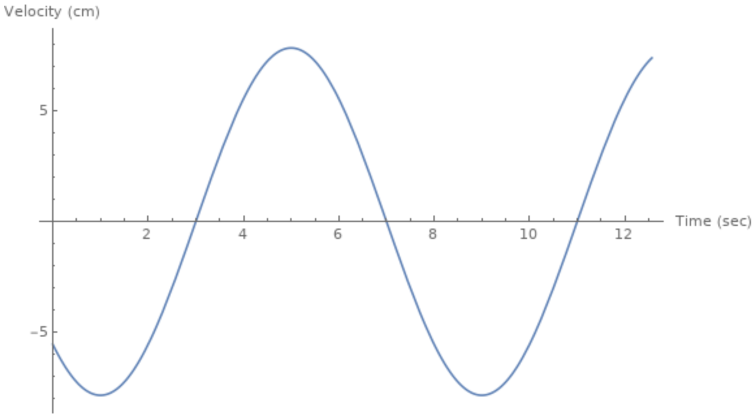
Problem 1 [5 points]

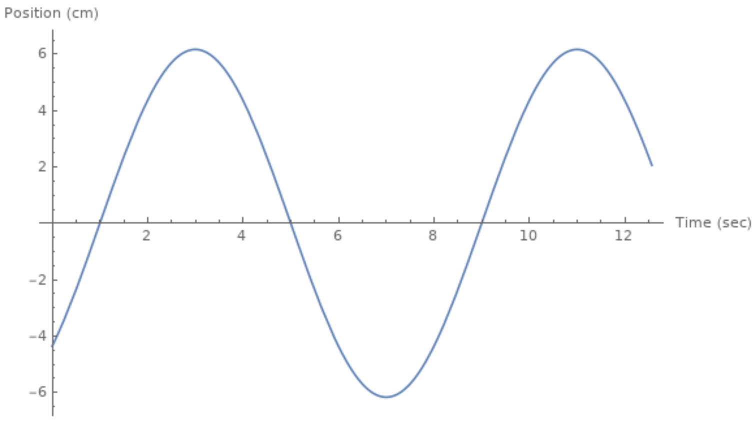
5 points for reading the note!

Problem 2 [10 points]

(a) $x(t) = A\cos(\omega t + \phi) = 10\cos(\pi t/4 + \pi/4)$
(b) $v(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t + \phi) = -\frac{5\pi}{2}\sin(\pi t/4 + \pi/4)$
(c) $a(t) = \frac{dv(t)}{dt} = -A\omega^2\cos(\omega t + \phi) = -\frac{5\pi^2}{8}\cos(\pi t/4 + \pi/4)$







The most common deductions here were for missing or incorrect units and axes labels (up to 3 points).

Problem 3 [10 points]

The resonant frequency of an LC circuit is given by

$$\omega = \sqrt{\frac{1}{LC}} \iff f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}. \tag{1}$$

This solution arises from Kirchoff's Loop Rule ($L\ddot{q} = -q/C$). Any incoming signals near this frequency–such as a radio station we want to tune into–will be preferentially amplified by the circuit. Thus, for optimal transmisison into our headphones, we rearrange this equation to find that the ideal capacitance would be

$$C = [(2\pi f)^2 L]^{-1} = [(2\pi(1.070 \times 10^6))^2(33 \times 10^{-6})]^{-1} = 6.7 \times 10^{-10} \text{ F}. \tag{2}$$

The most common deductions here were for confusing the cyclic frequency given (f) for ω (see clarification below).

Problem 4 [10 points]

Assuming the Earth is a perfect sphere of uniform density, Gauss's Law for gravitation states that the gravitational field a distance $r < R_\oplus$ from the center of the Earth depends only on the mass of the Earth contained within that radius r. Quantitatively,

$$g = \frac{GM_{\text{eff}}}{r^2}, \tag{3}$$

where the contained mass M_{eff} is given by

$$M_{\text{eff}}(r) = M_\oplus \cdot \frac{4\pi r^3/3}{4\pi R_\oplus^3/3} = M_\oplus \cdot (\frac{r}{R_\oplus})^3. \tag{4}$$

Thus, the gravitational acceleration is given by

$$g \equiv \ddot{r} = GM_\oplus R_\oplus^{-3} r, \tag{5}$$

and in particular, because gravity acts toward the center of the Earth, we have

$$\vec{g} \equiv \ddot{\vec{r}} = -GM_\oplus R_\oplus^{-3} \vec{r}. \tag{6}$$

The relationship is in the characteristic SHM form of $\ddot{\vec{r}} = -\omega^2 \vec{r}$, from which we can extract the frequency

$$\omega = \sqrt{GM_\oplus R_\oplus^{-3}} = \sqrt{(6.67 \times 10^{-11})(5.97 \times 10^{24})(6.37 \times 10^6)^{-3}} = 1.24 \times 10^{-3} s^{-1}, \tag{7}$$

or

$$f = \frac{\omega}{2\pi} = 1.97 \times 10^{-4} \text{ Hz}. \tag{8}$$

The most common deductions here were for assuming SHM instead of deriving it (see clarification of what SHM is below) and for finding ω instead of f as the question asked for.

Problem 5 [10 points]

(a) Two massless springs in series can be treated as a single spring with an effective spring constant

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}. \tag{9}$$

A simple and intuitive derivation comes from demanding that the net force on a point in between the two springs is zero (since a massless point experiencing a force would lead to nonphysical motion). Such a massless point in between the two springs feels a pull $k_1 x_1$ from spring 1 to the left and $k_2 x_2$ from the spring 2 to the right (in the configuration both springs are extended). Equating the forces, we find that $k_1 x_1 = k_2 x_2$. Now, we define an effective spring constant k_{eff} such that the force on the mass ($k_2 x_2$) can be given in the form of $k_{\text{eff}}(x_1 + x_2)$. Equating these formulas, we find that $k_{\text{eff}} = \frac{k_2 x_2}{x_1 + x_2} = \frac{k_2}{x_1/x_2 + 1} = \frac{k_2}{k_2/k_1 + 1} = \frac{k_1 k_2}{k_1 + k_2}$. The equation of motion for a mass on a spring is

$$F = m\ddot{x} = -kx \iff \ddot{x} = -\frac{k}{m}x, \tag{10}$$

from which we can extract the oscillation frequency

$$\omega = \sqrt{\frac{k}{m}}, \tag{11}$$

where (in our case) $k = k_{\text{eff}}$, so we have

$$\omega = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}. \tag{12}$$

(b) For each individual spring, $f_i \propto \omega_i \propto \sqrt{k_i}$, so it follows from equation (9) that

$$f_{\text{eff}}^2 = \frac{f_1^2 f_2^2}{f_1^2 + f_2^2}, \tag{13}$$

or

$$f_{\text{eff}} = \sqrt{\frac{f_1^2 f_2^2}{f_1^2 + f_2^2}} = \sqrt{\frac{1^2 \cdot 2^2}{1^2 + 2^2}} = 0.894 \text{ Hz}. \tag{14}$$

The most common deductions here were for taking the given frequencies f as angular frequencies ω (see below for the distinction).

Pointers & Clarifications

- **Cyclic and angular frequencies**
  - Angular frequency (denoted ω) measures the radians of oscillation covered per second and is typically expressed in units of s^{-1}. The angular frequency of a sinusoidal wave is the coefficient of the independent variable (i.e. time).
  - Cyclic frequency (denoted f) measures the cycles of oscillation completed per second, and to distinguish it from ω, we typically express its value in units of hertz (Hz). Since there are 2π radians in every cycle, the value of f is always a factor of 2π lower than that of ω when describing the same oscillation.
  - Because radians and cycles are unitless quantities, ω and f technically have the same physical units of s^{-1}. **The unit of Hz, however, specifically refers to "cycles per second"** and should thus be used when talking about cyclic frequencies for clarity.
- **Defining SHM.** Many of you intuitively predicted that oscillating through the Earth would exhibit simple harmonic motion; however, there is a distinction between oscillations and SHM. SHM specifically refers to motion resulting from forces proportional to the displacement of an object, while oscillations could occur when forces act with any magnitude in the opposite direction of the displacement. Thus, we could not know for sure that the motion in Problem 4 would be SHM until we arrived at equation (5) and saw that $\ddot{r} \propto -r$.