Physics 2a

Problem Set 2 SOLUTIONS

Problem 6. [10 points]

We will make liberal use of the small angle approximation here. When examining the system, it is important to note that the force in the direction tangent to the curve is not actually the same as the restoring force providing the oscillation. The restoring force points directly at the origin, which is the point we're restoring to, but the tangential force does not point at any specific location but changes as we move along the curve. However, since we are using small angles here, we can approximate the restoring force as the tangential force, and thus consider only the component of gravitational force pointing tangentially. The force of gravity is given by:

$$F = -mg. \tag{1}$$

and the tangent angle is given by:

$$\phi = \arctan(\frac{dy}{dx}), \tag{2}$$

the derivative gives the slope, and the arctan of the slope is the angle relative to horizontal. Then since we only want the force in the direction of the tangent, we take the force to be:

$$F = -mg\sin(\phi) = -mg\sin(\arctan(\frac{dy}{dx})). \tag{3}$$

Now we take $\frac{dy}{dx} = \sinh(\frac{x}{a})$. Then by drawing a triangle, we find:

$$\sin(\arctan(\frac{dy}{dx})) = \frac{\sinh(\frac{x}{a})}{\sqrt{1+\sinh^2(\frac{x}{a})}} = \frac{\sinh(\frac{x}{a})}{\cosh(\frac{x}{a})} = \tanh(\frac{x}{a}) \tag{4}$$

and

$$F = -mg\tanh(\frac{x}{a}). \tag{5}$$

Now we use the small angle approximation to approximate tanh. We find that:

$$\tanh(\frac{x}{a}) \approx \frac{x}{a} \tag{6}$$

by a first order Taylor approximation for tanh. So now we have

$$F = \frac{-mgx}{a} \tag{7}$$

which looks suspiciously like $F = -kx$ with $k = \frac{mg}{a}$.

Then we know for a SHO that

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{ma}} = \sqrt{\frac{g}{a}}. \tag{8}$$

It is also possible to solve this with forces directly at the origin, using components of both gravity and the normal force, or using a full energy approach $E = K + U$ and differentiating this to get the equation of motion (which you can learn in Ph 106a).

Problem 7. [10 points]
We know the pendulums are lightly damped, and therefore the equation of motion for each pendulum is

$$x(t) = Ae^{\frac{-\gamma t}{2}} \cos(\omega t), \tag{9}$$

where A is the initial amplitude, $\gamma = \frac{b}{m}$ with b the damping coefficient, and $\omega^2 = \frac{g}{l}$ with l the length.
Note that the damping coefficient is the same for both systems because we are given that the geometries are identical and the damping is only dependent on the shape of the bob and the medium in which the bob is swinging.
Now we know that we are only looking at the decay of the amplitude, so we can assume the cosine term to be 1 since that is its maximum. Thus the amplitude:

$$A(t) = Ae^{\frac{-\gamma t}{2}}. \tag{10}$$

We also want to solve for the time at which

$$A(t_{\frac{1}{2}}) = \frac{A}{2}, \tag{11}$$

because that is half the amplitude. So our equation to solve is

$$\frac{1}{2} = e^{\frac{-\gamma t_{\frac{1}{2}}}{2}}. \tag{12}$$

For the first pendulum, we plug in $t_{\frac{1}{2}} = t_A = 100s$ and $\gamma = \frac{b}{m_A}$ based on the values given in the problem to find

$$\frac{1}{2} = e^{\frac{-bt_A}{2m_A}}, \tag{13}$$

and for the second we plug in $t_B = 1000s$ to find:

$$\frac{1}{2} = e^{\frac{-bt_B}{2m_B}}. \tag{14}$$

Thus,

$$e^{\frac{-bt_A}{2m_A}} = e^{\frac{-bt_B}{2m_B}} \tag{15}$$

or

$$\frac{-bt_A}{2m_A} = \frac{-bt_B}{2m_B}. \tag{16}$$

Finally,

$$m_B = m_A \frac{t_B}{t_A} = 9kg \times \frac{1000s}{100s} = 90kg. \tag{17}$$

Problem 8. [10 points]

(a) We know that $Q = \frac{\omega_0}{\gamma}$ in general and that at critical damping $\gamma = 2\omega_0$. Then $Q = \frac{\omega_0}{2\omega_0} = \frac{1}{2}$.

(b) We know the equation of motion for a critically damped harmonic oscillator is

$$x(t) = (c_1 + c_2 t)e^{-\omega_0 t}. \tag{18}$$

Then we want to find the maximum velocity, and we are given a velocity-related initial condition, so we should get $v(t)$. By taking the derivative, we find

$$v(t) = -\omega_0 c_1 e^{-\omega_0 t} - \omega_0 c_2 t e^{-\omega_0 t} + c_2 e^{-\omega_0 t}. \tag{19}$$

We are given $v(0) = 0$ so

$$0 = -\omega_0 c_1 + c_2, \tag{20}$$

so

$$c_2 = \omega_0 c_1, \tag{21}$$

which means

$$v(t) = (-\omega_0 c_1 - \omega_0^2 c_1 t + \omega_0 c_1)e^{-\omega_0 t} = -\omega_0^2 c_1 t e^{-\omega_0 t}. \tag{22}$$

Now let's find the maximum by taking the derivative:

$$a(t) = (\omega_0^3 c_1 t - \omega_0^2 c_1)e^{-\omega_0 t}. \tag{23}$$

Set this equal to zero so

$$0 = (\omega_0^3 c_1 t - \omega_0^2 c_1)e^{-\omega_0 t} = (\omega_0 t - 1)e^{-\omega_0 t}, \tag{24}$$

which means

$$0 = \omega_0 t - 1, \tag{25}$$

so

$$t = \frac{1}{\omega_0}. \tag{26}$$

We are given

$$\omega_0 = 0.5 \frac{rad}{s}, \tag{27}$$

so

$$t = 2s. \tag{28}$$

Note that here the speed is maximized, so the acceleration is zero and so is the net force. So solving

$$-kx = b\frac{dx}{dt} \tag{29}$$

will work too, with b the damping coefficient.

Problem 9. [10 points]

(a) We'll first obtain the differential equation to be used in our circuit. By Kirchhoff's Loop Rule,

$$-V + IR + L\frac{dI}{dt} + \frac{q}{C} = 0. \tag{30}$$

If we differentiate with respect to t, we obtain:

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0 \tag{31}$$

or

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = 0. \tag{32}$$

Using the characteristic polynomial, we obtain two roots:

$$r_1 = \frac{-R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{33}$$

and

$$r_2 = \frac{-R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{34}$$

Please note r_1 and r_2 are both negative, and thus e^{r_1} and e^{r_2} will be decaying exponentials.
Now we can figure out the type of damping based on the discriminant. Evaluate:

$$\frac{R^2}{4L^2} - \frac{1}{LC} = \frac{500^2}{4 * 10^{-4}} - \frac{1}{10^{-8}} = 6.25 * 10^8 - 10^8 > 0. \tag{35}$$

Since the discriminant is positive, this is a case of heavy damping.
Also note that the discriminant being positive is equivalent to $\gamma > 2\omega_0$.

(b) Now we will use our roots r_1 and r_2 to obtain a solution to the differential equation. The general solution is

$$I(t) = Ae^{r_1 t} + Be^{r_2 t}. \tag{36}$$

We need some initial conditions to solve for A and B. When the switch is closed, the inductor resists any current change, and thus the voltage drop is exclusively over the inductor. Then the initial current

$$I(0) = 0. \tag{37}$$

We also notice that the voltage across the inductor is V, so since

$$V = L\dot{I}, \tag{38}$$

$$\dot{I}(0) = \frac{V}{L}. \tag{39}$$

So let's use the first initial condition. $I(0) = A + B = 0$ so $B = -A$. Then

$$I(t) = A(e^{r_1 t} - e^{r_2 t}), \tag{40}$$

so

$$\dot{I}(t) = A(r_1 e^{r_1 t} - r_2 e^{r_2 t}). \tag{41}$$

Using our second condition, we find

$$\dot{I}(0) = A(r_1 - r_2) = \frac{V}{L} \tag{42}$$

so

$$A = \frac{V}{L(r_1 - r_2)} \tag{43}$$

and

$$B = \frac{-V}{L(r_1 - r_2)}. \tag{44}$$

Thus, we have

$$I(t) = Ae^{r_1 t} + Be^{r_2 t} \tag{45}$$

with A and B as written above, and

$$q(t) = \int_0^t I(t) = \frac{A}{r_1}(e^{r_1 t} - 1) + \frac{B}{r_2}(e^{r_2 t} - 1). \tag{46}$$

Therefore,

$$V_c(t) = \frac{q(t)}{C} = \frac{V}{LC(r_1 - r_2)}(\frac{1}{r_1}(e^{r_1 t} - 1) - \frac{1}{r_2}(e^{r_2 t} - 1)). \tag{47}$$

As a check, we can evaluate

$$V_c(0) = \frac{V}{LC(r_1 - r_2}(0 - 0) = 0 \tag{48}$$

and

$$\begin{aligned} V_c(\infty) &= \frac{V}{LC(r_1 - r_2)}(\frac{-1}{r_1} - \frac{-1}{r_2}) \\ &= \frac{V}{LC(r_1 - r_2)} \frac{r_1 - r_2}{r_1 r_2} = \frac{V}{LCr_1 r_2} \\ &= \frac{V}{LC(\frac{R^2}{4L^2} - \frac{R^2}{4L^2} + \frac{1}{LC})} = V. \end{aligned} \tag{49}$$

These edge conditions are as expected.

The other way to do this problem is to solve the inhomogeneous differential equation $IR + L\frac{dI}{dt} + \frac{q}{C} = V$ or $\ddot{q}(t) + \frac{R}{L}\dot{q}(t) + \frac{1}{LC}q(t) = \frac{V}{L}$. We can use the same method as above to solve the homogeneous version, giving us the same roots r_1 and r_2, but we also must add a particular solution. Here it is convenient to note if we choose $q(t) = CV$ we find $\ddot{q} = 0$, $\dot{q} = 0$, and thus are left with $\frac{VC}{LC} = \frac{V}{L}$ which is clearly true. Then we simply must write out our homogeneous solution $q(t) = Ae^{r_1 t} + Be^{r_2 t} + VC$ and solve using initial conditions for $q(0)$ and $\dot{q}(0)$. Conveniently, the charge on the capacitor is zero at the start, and the initial current is zero by the logic above, so $q(0) = \dot{q}(0) = 0$. Thus we find $A + B + VC = 0$ and $r_1 A + r_2 B = 0$, so $B = -VC - A$, and $r_1 A - r_2 A = r_2 VC$ or $A = \frac{r_2 VC}{r_1 - r_2}$. Then $B = \frac{-r_1 VC}{r_1 - r_2}$. This yields an equation $q(t) = \frac{VC}{r_1 - r_2}(r_2 e^{r_1 t} - r_1 e^{r_2} t + r_1 - r_2) = \frac{VC}{r_1 - r_2}(r_2(e^{r_1 t} - 1) - r_1(e^{r_2 t} - 1))$. Recall from above $r_1 r_2 = \frac{1}{LC}$ and so we'll divide the exponential terms and multiply the leading coefficient by this. This gives $q(t) = \frac{V}{L(r_1 - r_2)}(\frac{1}{r_1}(e^{r_1 t} - 1) - \frac{1}{r_2}(e^{r_2 t} - 1))$ so $V_c(t) = \frac{V}{LC(r_1 - r_2)}(\frac{1}{r_1}(e^{r_1 t} - 1) - \frac{1}{r_2}(e^{r_2 t} - 1))$, which is identical to our first solution.

Problem 10. [10 points]

We have the same solution as in question 9, but now r_1 and r_2 are complex. Let's write

$$\alpha = \frac{R}{2L} \tag{50}$$

and

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \tag{51}$$

so that

$$r_1 = -\alpha + i\omega \tag{52}$$

and

$$r_2 = -\alpha - i\omega. \tag{53}$$

Then our general solution is

$$I(t) = e^{-\alpha t}(Asin(\omega t) + Bcos(\omega t)). \tag{54}$$

Using our initial conditions from before yields

$$I(0) = 0 = B \tag{55}$$

and

$$\dot{I}(t) = -\alpha e^{-\alpha t}Asin(\omega t) + \omega e^{-\alpha t}Acos(\omega t), \tag{56}$$

so

$$\dot{I}(0) = \frac{V}{L} = \omega A, \tag{57}$$

so

$$A = \frac{V}{\omega L}. \tag{58}$$

Then our solution is

$$I(t) = e^{-\alpha t}(\frac{V}{\omega L} sin(\omega t)). \tag{59}$$

We know $P = I^2 R$ so we can just plug in to get

$$P(t) = Re^{-2\alpha t}(\frac{V^2}{\omega^2 L^2} sin^2(\omega t)). \tag{60}$$

1 Comments

In general this was a well-done set. Problems 4 and 5 were particularly mathy, and people worked through them well and used the correct methods.

Problem 6: The only major problem here is that some people wrote ω in terms of x, and you can't write a constant in terms of a variable. I'd also like to note that the solution is identical to a pendulum of length a. This means the catenary should approximate a circle for small angles, and indeed it does (graph it)! The circle approximation is not nearly as good as the parabolic approximation given by a second order Taylor expansion around the minimum, but it is cool to see the power of the small angle approximation.

Problem 7: No real issues here.

Problem 8: No real issues here.

Problem 9: The really common errors were just algebra errors.

Problem 10: A common error was to use equation 2.28 from the book. This equation is derived from an RLC circuit without a DC voltage source, so it's a different system. The book's system has no battery and also has a charge on the capacitor initially. If you plug in the initial charge q_0 on the capacitor, you just get 0 in this system.