Problem 11 [10 points]

(a) 3 points

There are two different possible outcomes. If $-kx_0 < \mu_s m$, then the mass will not move since the static frictional force prevents the motion. If $-kx_0 > \mu_s m$, then the mass will begin to oscillate back and forth, each time the amplitude of the oscillation will shrink because of the frictional force.

- -1 point if no mention of the possibility of no motion $\mu_s m >$
- -1 point if no mention of oscillations, but includes frictional effect
- -1 point if no description of movement at all

(b) 3 points

The motion should stop in finite time, because the magnitude of the frictional force is not dependent on velocity or position and will thus not shrink. However, the magnitude of the restoring force is shrinking, so eventually the restoring force will be weaker than the frictional force which will cause the mass to cease motion. This could occur at either one of the turning points due to zero velocity - static friction or while it's moving due to kinetic friction. However, since the $\mu_s > \mu_k$, it will most likely stop at one of the turning points. This question can also be answered in terms of energy. Since the system begins with finite energy, and friction dissipates a constant amount each oscillation, it will clearly run out of energy in a finite time. The stopping condition in this case is not having enough energy for friction to be able to dissipate to complete the oscillation.

- -1 point if no mention of stopping condition
- -0.5 if not full explanation why it stops
- -2 points if wrong conclusion

(c) 4 points

Since the direction of the frictional will always appose the motion of the mass, we need to have two different equations of motion to describe the mass going in either direction. Thus, the two differential equations describing motion to the left and to the right are:

$$\ddot{x} + \mu_k + \omega_0^2 x = 0 \qquad \dot{x} > 0 \tag{1}$$

$$\ddot{x} - \mu_k + \omega_0^2 x = 0 \qquad \dot{x} < 0 \tag{2}$$

We can ansatz that the inhomogeneous solutions for each direction will be $x(t) = \frac{\alpha}{\omega_0^2}$ and $x(t) = -\frac{\alpha}{\omega_0^2}$. Plugging these in, it is clear that these work. We will then also add the homogeneous solution $x(t) = Ccos(\omega_0 t + \phi)$. A concise way of writing this can be done using the Heaviside or step function, as shown below:

$$x(t) = \begin{cases} C\cos(\omega_0 t + \phi) - \mu_k \omega_0^2 & \dot{x} > 0, \\ C\cos(\omega_0 t + \phi) + \mu_k \omega_0^2 & \dot{x} < 0 \end{cases} \tag{3}$$

This doesn't make perfect sense, since this does not account for what occurs at the turning points. Since the velocity is zero at those points, static frictional force should enter the equations. There is also a jump in position at the turning points as well due to the opposite signs on the friction terms. Finally, If we assume a constant C, we never see the decaying amplitude which we know must be occurring because of the friction. Hence, these equations are only valid from one turning point to the next and must be solved recursively.

- -2 points if forgot one of the directions on EOM
- -3 points if wrong EOM

- -1 point if forgot homogeneous solution
- -0.5 points if use μ_s instead of μ_k
- -2 points if doesn't explain whether the answer makes sense
- -0 points if didn't describe what happened at $\dot{x} = 0$

Problem 12 [8 points]

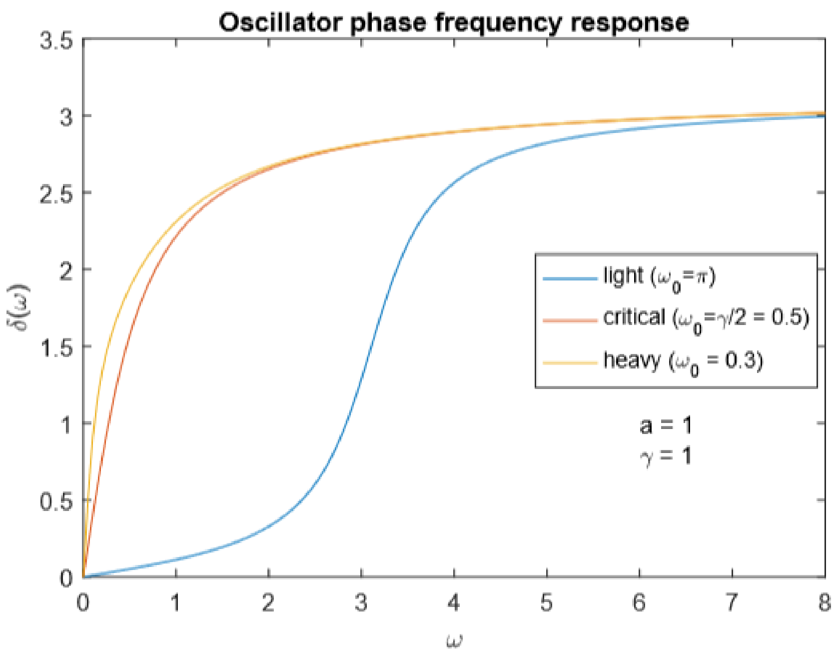
The hard part of this problem is the difficult algebra to get to a very satisfying answer. Given that $P(t) = R(CV\frac{\omega_0^2}{\omega})^2 e^{-\gamma t}\sin^2\omega t$ where $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}, \omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$, we know that to find the total energy dissipated from the system, we must integrate P(t) from t = 0 to infinity.

$$
\begin{aligned}
E_{tot} = \int_0^\infty P(t)dt = \int_0^\infty R(CV\frac{\omega_0^2}{\omega})^2 e^{-\gamma t} sin^2(\omega t)dt \\
= \frac{R}{2}(CV\frac{\omega_0^2}{\omega})^2 \int_0^\infty e^{-\gamma t}(1 - cos(2\omega t))dt \\
= \frac{R}{2}(CV\frac{\omega_0^2}{\omega})^2 \int_0^\infty e^{-\gamma t}(1 - \frac{1}{2}(e^{2i\omega t} + e^{-2i\omega t}))dt \\
= -\frac{R}{2}(CV\frac{\omega_0^2}{\omega})^2 e^{-\gamma t}(\gamma + \frac{1}{2}((-\gamma + 2i\omega)e^{2i\omega t} - (\gamma + 2i\omega)e^{-2i\omega t})\Big|_0^\infty \\
= -\frac{R}{2}(CV\frac{\omega_0^2}{\omega})^2(0 - \frac{1}{\gamma} - \frac{1}{2}(\frac{1}{(-\gamma + i\omega)} - \frac{1}{(\gamma + i\omega)})) \\
= -\frac{R}{2}(CV\frac{\omega_0^2}{\omega})^2 \frac{4\omega^2}{\gamma^3 + 4\gamma\omega^2} = -RC^2V^2\frac{2\omega_0^4}{\gamma^3 + 4\gamma\omega^2} = CV^2\frac{2\omega_0^2}{\gamma^2 + 4\omega^2} = \frac{CV^2}{2} \quad (4)
\end{aligned}
$$

We notice that we get the capacitor energy as the answer. Thus, the total energy dissipated is used to move the charge (CV) on the capacitor against the resistor.

- -1 for not simplifying fully/ not plugging in $\omega_0, \omega, or \gamma$
- -1 for small error in algebra
- -2 for small conceptual error
- -4 for wrong definition of total energy/wrong bounds

Problem 13 [5 points]



Using the shifted definition of $\tan\delta$ that is more natural in this question, we get that the oscillator phase frequency response is $\tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}$. We found this using $\sin\delta = \frac{\gamma\omega A}{F_0/m}$ and $\cos\delta = \frac{(\omega_0^2-\omega^2)A}{F_0/m}$.

- -1 points if equations aren't given
- -1 if graphs are wrong but equations are right
- -3 if graphs and equations are wrong
- -0 if graphs do not have shifted definition of arctan

Problem 14 [7 points]

(a) 4 points

We first must find the equation of motion of the system and then we can get those values from that equation. We will solve from $Q = CV$, $V = IR$ and $V = \frac{dI_L}{dt}$ to find the EOM in terms of V.

$$Q = CV \rightarrow I_C = \frac{dQ}{dt} \rightarrow I_C = C\frac{dV}{dt} \rightarrow \frac{dI_C}{dt} = C\frac{d^2V}{dt^2} \tag{5}$$

$$V = IR \rightarrow I_R = C\frac{V}{R} \rightarrow \frac{dI_R}{dt} = C\frac{dV}{dt} \tag{6}$$

$$\frac{dI_L}{dt} = \frac{V}{L} \tag{7}$$

From the junction rule, we know $I_0\cos\omega t = I_C + I_L + I_R \rightarrow -\omega I_0 \sin\omega t = \frac{dI_C}{dt} + \frac{dI_L}{dt} + \frac{dI_R}{dt}$. From that, we can get our equation of motion:

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = -\frac{\omega}{C}I_0\sin\omega t \tag{8}$$

The coefficient of your V term will be the natural frequency ω_0 squared: $\omega_0^2 = \frac{1}{LC}$. We can also find Q by the relationship $Q = \frac{\omega_0}{\gamma} = R\sqrt{\frac{C}{L}}$ where $\gamma = \frac{1}{RC}$.

(b) 3 points

The long term behavior or the steady state solution of the voltage (which is the same across any of the circuit elements due to them being in parallel), once the natural oscillations have died out due to damping can be seen to be the inhomogenous solution of the differential equation for V(t) above and is given by,

$$V_{steady}(t) = V_0(\omega)\cos(\omega t - \delta), \tag{9}$$

where $\delta(\omega)$ is the appropriately defined steady state phase difference and the amplitude $V_0(\omega)$ is seen to be (on the same lines as how we solved for the steady state solutions for driven oscillations),

$$V_0(\omega) = \frac{I_0\omega/C}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} \tag{10}$$

where as we solved in part (a), $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{1}{RC}$. Now at resonance, $\omega = \omega_0$, we can find the peak voltage $V_0(\omega = \omega_0)$ and it can easily be seen to be,

$$V_0 = I_0 R, \tag{11}$$

which is precisely what it would be had the circuit had no capacitor or inductor! Indeed resonance is defined by that frequency at which the circuit looks purely resistive. Said differently, as ew know at resonance, there is maximum power transfer from the driving agent and since in RLC circuits, only the resistor can dissipate power, hence the circuit works to behave resistively so as to maximize power transfer.

- -1 point each for wrong ω_0 and Q
- -1 point for not deriving the correct ω_0 and Q
- -1 incomplete answer for 4.2
- -1 if no comparison to circuit without L and C

Common Issues

- Problem 11a: Many people forgot to mention that if $\mu_s m > k|x_0|$, then the mass will not move.
- Problem 11b: Almost everyone got the stopping condition, but some people didn't prove that it was actually reachable. You should mention that the amplitude decreases with each oscillation, or that energy is being dissipated at a continuous rate; therefore, there will come a time when the amplitude has been reduced far enough to satisfy the static stopping condition.
- Problem 12: Many people forgot to plug in ω_0, ω, and γ to simplify the equation.
- Problem 13: Many people forgot to write out the equation that they were plotting. If the software showed the equation clearly, with the correct variable names, I did not count anything off.
- Problem 14a: Some people just stated what ω_0 and γ were without doing the circuit calculation, but then often used the wrong value of γ.