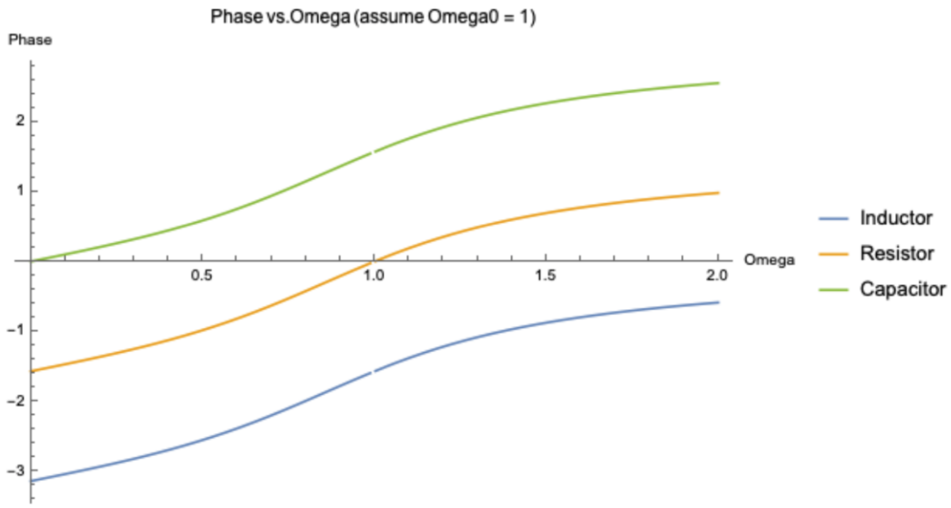
Problem 15 [10 points]

a) We'd like to solve for the voltages across all three components. To do this, we start with Kirchhoff's Loop Rule, which relates all of the voltages in the problem. Writing in complex notation gives $V_0e^{i\omega t}) = L\ddot{q} + R\dot{q} + \frac{1}{C}q$. Since we only care about the steady state solution, we don't need to find the solutions to the homogeneous equation, as those will have decay terms. We only need the particular solution. Thus we can guess an exponential form $q(t) = Ae^{i\omega t}$. Plugging this in and canceling the exponentials yields $-\omega^2 LA + i\omega RA + \frac{1}{C}A = V_0$, so $A = \frac{V_0}{L} * \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$, where $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$. Then $q(t) = Re(\frac{V_0/L}{\omega_0^2-\omega^2+i\omega\gamma}e^{i\omega t})$. Now if we write $\omega_0^2 - \omega^2 + i\omega\gamma$ in the form $C(\cos\phi + i\sin\phi)$, we find that $C = \sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}$ and $\phi = \arctan\frac{\omega\gamma}{\omega_0^2-\omega^2}$. Then we can write $q(t) = Re(\frac{V_0/L}{\omega_0^2-\omega^2+i\omega\gamma}e^{i\omega t}) = Re(\frac{V_0}{L}\frac{e^{-i\phi}}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}e^{i\omega t}) = \frac{V_0/L}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\cos(\omega t - \phi)$. Then we can calculate all of the relevant voltages. $V_C = \frac{q(t)}{C} = \frac{V_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\cos(\omega t-\phi)$. $V_R = R\dot{q} = \frac{-V_0\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\sin(\omega t-\phi)$. $V_L = L\ddot{q} = \frac{-V_0\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\cos(\omega t-\phi)$. But now we note that $-\sin\theta = \cos(\theta + \frac{\pi}{2})$ and $-\cos\theta = \cos(\theta+\pi)$. Then the relative phases are ϕ for the capacitor, $\phi - \frac{pi}{2}$ for the resistor, and $\phi - \pi$ for the inductor. We can plot these as follows.



b) Let's take limits as $\omega \to 0$. For the capacitor, we see $V_C \to \frac{V_0\omega_0^2}{\omega_0^2}\cos\phi = V_0$. For the resistor, we see $V_R \to \frac{-V_0\gamma\omega}{\omega_0^2}\sin\phi = 0$ because $\omega = 0$. For the inductor, we see $V_L \to \frac{-V_0\omega^2}{\omega_0^2}\cos\phi = 0$ because $\omega = 0$. Then all of the voltage drop is over the capacitor, which makes sense because the $\omega = 0$ limit is a DC case, and the steady state is simply a fully charged capacitor. As $\omega \to \infty$, we see $V_C \to \frac{V_0\omega_0^2}{\omega^2}\cos(\omega t - \phi) \to 0$ because we have a large power of ω in the denominator. We see $V_R \to \frac{-V_0\gamma\omega}{\omega^2}\sin(\omega t-\phi) \to 0$ because the power of ω in the denominator is larger than that in the numerator. We see $V_L \to \frac{V_0\omega^2}{\omega^2}\cos(\omega t - \phi + \pi)$ so the amplitude goes to V_0. This makes sense because as frequency gets very high, the current is constantly changing, so the inductor is constantly resisting these changes. Thus the voltage drop over the inductor dominates.

- 3 points for appropriate derivation of voltage behavior across inductor, capacitor, and resistor
- 2 points for graph depicting phase difference
- 2 points each for correct behavior of V_c and V_L at $\omega \to 0$ and $\omega \to \infty$
- 1 point to behavior of V_R

Problem 16 [10 points]

We've already calculated the voltage drop over the resistor from problem 16. We have $\frac{-V_0\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\sin(\omega t-\phi)$. We know that the power dissipated across a resistor is given by $\frac{V_R^2}{R}$, so we want to find where V_R is at a maximum. This occurs when $\frac{V_0\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}$ is at a maximum, since the oscillating term is time dependent and thus will reach 1 at some time. We can take a derivative with respect to ω to find this maximum. $\frac{dV_R}{d\omega} = \frac{V_0\gamma\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2} - V_0\gamma\omega\frac{-2\omega_0^2\omega+2\omega^3+\omega\gamma^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}$. Setting equal to zero, we can see that our maximum is when $\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2} = \omega\frac{-2\omega_0^2\omega+2\omega^3+\omega\gamma^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}$ or $(\omega_0^2-\omega^2)^2+(\omega\gamma)^2 = -2\omega_0^2\omega^2+2\omega^4+(\omega\gamma)^2$ or $\omega_0^4+\omega^4-2\omega_0^2\omega^2 = -2\omega_0^2\omega^2+2\omega^4$ or $\omega_0^4=\omega^4$ or $\omega_0=\omega$. Then we will calculate the power dissipated at resonance. At resonance, $\frac{-V_0\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}\sin(\omega t-\phi) = -V_0\sin(\omega t-\phi)$ so the magnitude is just V_0. Then the power dissipated is $P_{peak} = \frac{V_0^2}{R} = 2.5W$. This is larger than the rating on the resistor, so the resistor will not work for this case. However, sometimes resistors are rated for average power rather than peak power. To find the average power, we can see that the power will oscillate as sine squared, because $P(t) = \frac{V_0^2\sin^2(\omega t-\phi)}{R}$. The average value of sine squared is $\frac{1}{2}$, so we must divide P_{peak} by 2. This yields $P_{avg} = 1.25W < 2W$ so the resistor will be sufficient in this case.

- 5 points for considering the right voltage and power expression
- 3 points for considering the average power
- 2 points for appropriate conclusion from results

Problem 17 [10 points]

For small oscillations, we know $F = -\frac{mg}{l}x$ for a pendulum. Then, adding in the spring between, we can write the equations of motion for the two springs. $m_a\ddot{x_a} = -\frac{m_a g}{l}x_a - k(x_a-x_b)$ and $m_b\ddot{x_b} = -\frac{m_b g}{l} + k(x_a-x_b)$. We want to write this in the form $\ddot{x}+Mx=0$ to solve for the eigenvectors of M, which are the normal modes. Using our equations of motion, we can write $M = \begin{pmatrix} \frac{g}{l}+\frac{k}{m_a} & \frac{-k}{m_a} \\ \frac{-k}{m_b} & \frac{g}{l}+\frac{k}{m_b} \end{pmatrix}$. Solving for the eigenvalues of this matrix yields the quadratic equation $\lambda^2 - (\frac{2g}{l}+\frac{k}{m_a}+\frac{k}{m_b})\lambda + (\frac{g}{l})^2 + \frac{g}{l}(\frac{k}{m_a}+\frac{k}{m_b}) = 0$, which has a solution

$$\begin{aligned}\lambda &= \frac{1}{2}(\frac{2g}{l} + k\frac{m_a+m_b}{m_am_b} \pm \sqrt{4(\frac{g}{l})^2 + 4(\frac{g}{l})k(\frac{m_a+m_b}{m_am_b}) + k^2(\frac{m_a+m_b}{m_am_b})^2 - 4(\frac{g}{l})^2 - 4(\frac{g}{l})k(\frac{m_a+m_b}{m_am_b})}) \\ &= \frac{1}{2}(\frac{2g}{l} + k\frac{m_a+m_b}{m_am_b} \pm k(\frac{m_a+m_b}{m_am_b})).\end{aligned}$$

So the two solutions are $\frac{g}{l}$ and $\frac{g}{l}+k\frac{m_a+m_b}{m_am_b}$. Now note that $Mv=\lambda v$, so we can write $\ddot{x}+\lambda x=0$, which implies $\lambda=\omega^2$. Then $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l}+k\frac{m_a+m_b}{m_am_b}}$. In the limit that $m_a=m_b$, we find that ω_2 becomes $\sqrt{\frac{g}{l}+\frac{2k}{m}}$ which is what was derived in lecture.

- 2 pts for EOM
- 4 pts for solving eigenvalue problem
- 2 pts for correct answer for frequencies
- 2 pts for discussing limit of two equal masses

Problem 18 [10 points]

We have from lecture the normal modes as $v_1 = \frac{\sqrt{2}}{2}(x_a-x_b)$, $\omega_1 = \sqrt{\frac{g}{l}+\frac{2k}{m}}$, and $v_2 = \frac{\sqrt{2}}{2}(x_a-x_b)$, $\omega_2 = \frac{g}{l}$. We know each normal mode has the equation of motion of a simple harmonic oscillator, so we have $v_1(t) = A_1\cos(\omega_1 t+\phi_1)$ and $v_2(t) = A_2\cos(\omega_2 t+\phi_2)$. We can invert normal coordinates to get $x_a = \frac{\sqrt{2}}{2}(A_1\cos(\omega_1 t+\phi_1) + A_2\cos(\omega_2 t+\phi_2))$ and $x_b = \frac{\sqrt{2}}{2}(A_2\cos(\omega_2 t+\phi_2) - A_1\cos(\omega_1 t+\phi_1))$. Now we can write the initial condition equations. $x_a(0) = A = \frac{\sqrt{2}}{2}(A_1\cos\phi_1 + A_2\cos\phi_2)$, $x_b(0) = 0 = \frac{\sqrt{2}}{2}(A_2\cos\phi_2 - A_1\cos\phi_1)$, $\dot{x_a}(0) = 0 = \frac{\sqrt{2}}{2}(-A_1\omega_1\sin\phi_1 - A_2\omega_2\sin\phi_2)$, $\dot{x_b}(0) = 0 = \frac{\sqrt{2}}{2}(-A_2\omega_2\sin\phi_2 + A_1\omega_1\sin\phi_1)$. So $\dot{x_a}(0) = \dot{x_b}(0)$ implies $-A_1\sin\phi_1 = A_1\sin\phi_1$ and $\dot{x_a}(0) = -\dot{x_b}(0)$ implies $-A_2\sin\phi_2 = A_2\sin\phi_2$. But since we are not in either normal mode because of the initial conditions, we know $A_1 \neq 0$ and $A_2 \neq 0$, so $\phi_1 = \phi_2 = 0$. So we can rewrite the initial conditions as $x_a(0) = A = \frac{\sqrt{2}}{2}(A_1 + A_2)$ and $x_b(0) = 0 = A_2 - A_1$. Then $A_1 = A_2$ and $A = A_1\sqrt{2}$ which means our equation for x_b becomes $x_b = \frac{A}{2}(\cos(\omega_2 t) - \cos(\omega_1 t))$. For x_b to reach A, we require $\cos(\omega_2 t) = 1$ and $\cos(\omega_1 t) = -1$, so $\omega_2 t = 2n\pi$ for n a natural number and $\omega_1 t = (2m-1)\pi$ for m a natural number. Then we need $\frac{\omega_1}{\omega_2} = \frac{2m-1}{2n}$, so $\frac{\omega_1}{\omega_2}$ must be a rational number with an odd numerator and an even denominator.

- 6 pts for correct $x_b(t)$
- 1 pt for recognizing that one term must be 1 while other must be -1
- 3 pts for correct condition

Problem 19 [10 points]

a) We can start with the restoring force in the vertical direction, which is $F = -T\sin\theta_1 - T\sin\theta_2$. For small displacements, we can approximate $\sin\theta \approx \frac{\Delta y}{l}$, so we get $F = -\frac{T}{l}(y_i - y_{i-1}) + \frac{T}{l}(y_{i+1} - y_i)$ or $m\ddot{y_i} = \frac{T}{l}(y_{i-1} - 2y_i + y_{i+1})$. Now we remember the edges are such that $y_0 = 0$ and $y_{n+1} = 0$, so the general differential equation is still $\ddot{y_i} = \frac{T}{ml}(y_{i-1} - 2y_i + y_{i+1})$, but for $i = 1$, $\ddot{y_1} = \frac{T}{ml}(-2y_1 + y_2)$, and for $i = n$, $\ddot{y_n} = \frac{T}{ml}(y_{n-1} - 2y_n)$.

- 2 pts for setup
- 2 pts for small angle approximation
- 2 pts for correct answer

b) We want this in the form $\ddot{y} + My = 0$. The first row uses our equation for y_1, so it's $\frac{T}{ml}(2 - 10...0)$ and then the last row uses our equation for y_n, so it's $\frac{T}{ml}(0...0, -1, 2)$. All the other rows use our middle equation, so the matrix looks like $M = \frac{T}{ml}\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ 0 & -1 & 2 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 2 & -1 & 0 \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ 0 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$.

- 2 pts for correct differential equation
- 2 pts for correct matrix

Comments

Problem 15: The most common problem here was that people did not take into account the phase difference between the driving force and the components of the circuit. It's important to realize that that phase difference varies with ω. Additionally, a good check on your results for part (b) would be to plug in those results to Kirchhoff's Loop Rule to confirm your result is physical and we are interested in the amplitude of each value.

Problem 16: Just a reminder to justify your answers here. It's important to tell why the peak frequency is what it is. Also as suggested by the problem, we are mainly concerned of average power instead of the instantaneous maximum power.

Problem 17: There weren't any significant issues here.

Problem 18: The most common problems were swapping ω_1 and ω_2 resulting in the wrong conclusions. The solution to $\cos(\theta) = -1$ is $\theta = n\pi$ for *odd* integers n, not just any integers (or equivalently, $\theta = (2m+1)\pi$

for any integer m).

Problem 19: There weren't any significant issues here. As always, remember to watch your signs!

Overall: I've taken off 1 to 2 points for minor mistakes including algebra, depending on demonstration of correct understanding and/or severity of error.
